Taiji–Skyrme Lagrangian and Hamiltonian Framework

Internal Mathematical Draft

Abstract

In this document we formulate the Skyrme–type version of the Taiji model. The two Taiji fields Ψ (Potential Field) and Φ (Excitation) are promoted to a unified SU(2)–valued field $U(x)$ built from a Taiji spinor doublet. We derive the Lagrangian density, its time–space decomposition, the canonical Hamiltonian density, and the symmetric energy–momentum tensor. The static limit reproduces the usual Skyrmion energy functional and provides the foundation for gravitating Taiji knots.

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1 Field Content and Kinematic Structure

1.1 Taiji Spinor and SU(2) Field

We start from the two complex Taiji fields

$$\Psi(x),\ \Phi(x) \in \mathbb{C}, \tag{1}$$

and assemble them into a Taiji spinor doublet

$$\mathcal{Z}(x) \equiv \begin{pmatrix} \Psi(x) \\ \Phi(x) \end{pmatrix}, \qquad \mathcal{Z}(x) \in \mathbb{C}^2. \tag{2}$$

The total field magnitude is

$$|\mathcal{Z}(x)|^2 = \mathcal{Z}^\dagger(x)\,\mathcal{Z}(x) = |\Psi(x)|^2 + |\Phi(x)|^2. \tag{3}$$

In the spontaneously broken phase we fix a nonzero vacuum magnitude

$$|\mathcal{Z}(x)|^2 \to v^2 \quad \text{in the vacuum}, \qquad v > 0, \tag{4}$$

and introduce the normalized spinor

$$\widehat{\mathcal{Z}}(x) \equiv \frac{\mathcal{Z}(x)}{|\mathcal{Z}(x)|}, \qquad \widehat{\mathcal{Z}}^\dagger \widehat{\mathcal{Z}} = 1. \tag{5}$$

The normalized spinor $\widehat{\mathcal{Z}}$ parametrizes a point on the three–sphere S^3. It can be mapped to an SU(2)–valued field

$$U(x) \in \mathrm{SU}(2), \qquad U^\dagger U = \mathbf{1}, \qquad \det U = 1. \tag{6}$$

The explicit map from $\widehat{\mathcal{Z}}$ to U is not required for the derivations below; only the SU(2) structure and unitarity are needed.

1.2 Right–Invariant Currents

It is convenient to work with the right–invariant current

$$R_\mu(x) \equiv U^\dagger(x)\,\partial_\mu U(x), \qquad \mu = 0, 1, 2, 3. \tag{7}$$

The current R_μ takes values in the Lie algebra $\mathfrak{su}(2)$ and satisfies

$$R_\mu^\dagger = -R_\mu, \tag{8}$$

so it is anti–Hermitian. Indices are raised and lowered with the Minkowski metric

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1), \qquad R^\mu = \eta^{\mu\nu} R_\nu. \tag{9}$$

Throughout this document we use the trace in the fundamental representation, normalized such that

$$\mathrm{Tr}(\sigma_a \sigma_b) = 2\,\delta_{ab}, \tag{10}$$

where σ_a are the Pauli matrices.

2 Skyrme–Taiji Lagrangian

2.1 Action and Decomposition

We work in flat Minkowski spacetime with metric $\eta_{\mu\nu}$. The Skyrme–type Taiji Lagrangian density is chosen as

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 - V(U), \tag{11}$$

where

$$\mathcal{L}_2 = \frac{f^2}{16}\,\mathrm{Tr}\big(R_\mu R^\mu\big), \tag{12}$$

$$\mathcal{L}_4 = \frac{1}{32e^2}\,\mathrm{Tr}\big([R_\mu, R_\nu][R^\mu, R^\nu]\big), \tag{13}$$

and the potential $V(U)$ introduces a mass scale:

$$V(U) = \frac{m^2 f^2}{4}\,\mathrm{Tr}\big(\mathbf{1} - U\big). \tag{14}$$

Here:

- f is a characteristic energy scale, identified with the vacuum magnitude of the Taiji spinor, $f \equiv v$;
- e is a dimensionless coupling constant;
- m is a mass parameter controlling the exponential decay of the Skyrmion tail.

The quadratic term $\mathcal{L}_2$ provides the usual nonlinear σ–model kinetic term, while $\mathcal{L}_4$ is the Skyrme quartic term. The potential $V(U)$ softly breaks chiral symmetry and selects $U = \mathbf{1}$ as the vacuum configuration.

2.2 Coherence Spinor and Covariant Coupling to the Internal Connection

The Taiji spinor $Z(x)$ and its unit–norm representative $\hat{Z}(x)$ were introduced in the general mathematical framework. Whenever the coherence amplitude $C(x) = Z^\dagger Z$ is nonzero, the normalised spinor $\hat{Z} = Z/\sqrt{C}$ satisfies $\hat{Z}^\dagger \hat{Z} = 1$ and parametrises an internal coherence state. As discussed there, the replacement

$$\hat{Z}(x) \;\longrightarrow\; e^{i\lambda(x)}\hat{Z}(x), \qquad \lambda(x) \in \mathbb{R}, \tag{15}$$

leaves both $\hat{Z}^\dagger \hat{Z}$ and the associated $SU(2)$ field $U(x)$ unchanged. The physically relevant internal state is therefore the ray $[\hat{Z}]$, and the transformation (15) is interpreted as a local $U(1)$ gauge freedom in the coherence sector.

To track how the coherence state twists in internal space, we introduce a connection one–form $A_\mu(x)$ with the $U(1)$ gauge transformation law

$$A_\mu(x) \;\longrightarrow\; A'_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x). \tag{16}$$

At this stage we treat A_μ as an independent field that will furnish the electromagnetic sector. The interaction between the coherence spinor and this connection is encoded in the covariant derivative

$$D_\mu \hat{Z} \equiv (\partial_\mu - iqA_\mu)\,\hat{Z}, \tag{17}$$

where q is an effective coherence charge. Under the gauge transformation (15)–(16), the covariant derivative transforms homogeneously,

$$\hat{Z}(x) \to e^{i\lambda(x)}\hat{Z}(x) \quad \Rightarrow \quad D_\mu \hat{Z}(x) \to e^{i\lambda(x)} D_\mu \hat{Z}(x), \tag{18}$$

so that the scalar quantity $(D_\mu \hat{Z})^\dagger D^\mu \hat{Z}$ is strictly gauge invariant.

A minimal gauge–invariant kinetic term for the coherence spinor is then

$$\mathcal{L}_{\hat{Z},A} = \kappa\,(D_\mu \hat{Z})^\dagger D^\mu \hat{Z} - V_{\hat{Z}}(\hat{Z}), \tag{19}$$

with κ a real positive constant and $V_{\hat{Z}}$ an effective potential compatible with the underlying coherence structure. The full Lagrangian studied in this document may thus be extended schematically to

$$\mathcal{L}_{\text{total}} = \mathcal{L}_2 + \mathcal{L}_4 - V(U) + \mathcal{L}_{\hat{Z},A} + \mathcal{L}_{\text{EM}}, \tag{20}$$

where $\mathcal{L}_{\text{EM}}$ denotes the free Maxwell sector defined in the electromagnetic subsection and $\mathcal{L}_2, \mathcal{L}_4, V(U)$ are the standard Skyrme contributions.

2.3 Explicit Time–Space Split

Using the Minkowski metric, we have

$$R_\mu R^\mu = R_0 R^0 + R_i R^i = R_0 R_0 - \sum_{i=1}^{3} R_i R_i, \tag{21}$$

so the quadratic term becomes

$$\mathcal{L}_2 = \frac{f^2}{16}\left[\mathrm{Tr}\big(R_0 R_0\big) - \sum_{i=1}^{3} \mathrm{Tr}\big(R_i R_i\big)\right]. \tag{22}$$

The Skyrme term can be written as

$$\mathcal{L}_4 = \frac{1}{32e^2}\,\mathrm{Tr}\big([R_\mu, R_\nu][R^\mu, R^\nu]\big). \tag{23}$$

Upon splitting time and space indices, using $\eta^{00} = +1$ and $\eta^{ij} = -\delta^{ij}$, one obtains the familiar structure

$$\mathcal{L}_4 = \frac{1}{16e^2}\,\mathrm{Tr}\big([R_0, R_i][R_0, R_i]\big) - \frac{1}{32e^2}\,\mathrm{Tr}\big([R_i, R_j][R_i, R_j]\big), \tag{24}$$

where repeated spatial indices $i, j = 1, 2, 3$ are summed over.

The mass potential $V(U)$ does not contain derivatives of U and carries no explicit metric dependence beyond the overall factor in the action.

2.4 Electromagnetic Sector as a Coherence Connection

The Taiji spinor doublet $Z(x)$ and its normalized version $\hat{Z}(x)$ were introduced in Section 1.1. As discussed there and in the separate mathematical framework document, $\hat{Z}(x)$ admits a local phase redundancy

$$\hat{Z}(x) \;\longrightarrow\; e^{i\lambda(x)}\hat{Z}(x),$$

under which all physical quantities built from the associated $SU(2)$ field $U(x)$ remain invariant. This $U(1)$ redundancy permits the definition of a composite internal connection

$$A_\mu(x) \equiv -\,i\,\hat{Z}^\dagger(x)\,\partial_\mu \hat{Z}(x), \qquad \mu = 0, 1, 2, 3, \tag{25}$$

which transforms as

$$A_\mu(x) \;\longrightarrow\; A'_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x) \tag{26}$$

under local phase rotations. This is precisely the $U(1)$ gauge transformation law characteristic of an electromagnetic potential.

In the present document we do not attempt to derive a Maxwell kinetic term from the Skyrme sector. Instead, we introduce a minimal electromagnetic Lagrangian density for the connection A_μ,

$$\mathcal{L}_{\rm EM} = -\frac{1}{4g_{\rm EM}^2} F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{27}$$

where $g_{\rm EM}$ is a dimensionless coupling and the Minkowski metric is used to raise indices. The field strength $F_{\mu\nu}$ is manifestly invariant under (26), so $\mathcal{L}_{\rm EM}$ is locally $U(1)$ gauge invariant.

The full Skyrme–Taiji Lagrangian is thus extended to

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 - V(U) + \mathcal{L}_{\rm EM}, \tag{28}$$

with the understanding that A_μ is interpreted as the dynamical connection associated with the internal coherence phase of the Taiji spinor.

The full Skyrme–Taiji Lagrangian is thus extended to

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 - V(U) + \mathcal{L}_{\rm EM}, \tag{29}$$

with the understanding that A_μ will be interpreted as the dynamical connection associated with the internal coherence phase of the Taiji spinor. At this stage the electromagnetic sector is "free" (no explicit coupling to additional charged matter fields), but it already supports propagating Maxwell waves (photons) in the usual way.

3 Canonical Structure and Hamiltonian Density

3.1 General Remarks

The field $U(x)$ takes values in SU(2), so a fully explicit canonical formalism requires a choice of coordinates on the group manifold or an equivalent parameterization in terms of real fields with constraints. For the purposes of energy and gravitational coupling, it is sufficient to express the Hamiltonian density in terms of the currents R_μ.

The Hamiltonian density $\mathcal{H}$ is defined as

$$\mathcal{H} = \Pi : \partial_0 U + \Pi^\dagger : \partial_0 U^\dagger - \mathcal{L}, \tag{30}$$

where Π denotes the matrix of canonical momenta conjugate to U, and $A : B$ denotes a suitable trace pairing over internal indices. The detailed form of Π depends on the chosen parameterization of U. However, a compact and standard expression for $\mathcal{H}$ can be obtained directly in terms of R_0 and R_i.

3.2 Hamiltonian Density in Terms of Currents

Using the Skyrme form of the Lagrangian, one finds that the Hamiltonian density can be written as

$$\mathcal{H} = \frac{f^2}{16}\,\mathrm{Tr}(R_0R_0) + \frac{f^2}{16}\,\mathrm{Tr}(R_iR_i) + \frac{1}{16e^2}\,\mathrm{Tr}\big([R_0,R_i][R_0,R_i]\big) + \frac{1}{32e^2}\,\mathrm{Tr}\big([R_i,R_j][R_i,R_j]\big) + V(U). \tag{31}$$

In this expression:

- the terms involving R_0 correspond to kinetic energy;
- the terms involving R_i and their commutators correspond to spatial gradient energy and torsional stiffness;

- $V(U)$ is the potential energy density.

The total Hamiltonian is

$$H = \int d^3x\, \mathcal{H}(t, \mathbf{x}). \tag{32}$$

3.3 Static Limit

For static configurations, U is independent of time:

$$\partial_0 U = 0 \quad \Rightarrow \quad R_0 = U^\dagger \partial_0 U = 0. \tag{33}$$

Then the Hamiltonian density reduces to

$$\mathcal{H}_{\text{static}} = \frac{f^2}{16}\,\mathrm{Tr}(R_i R_i) + \frac{1}{32e^2}\,\mathrm{Tr}\big([R_i, R_j][R_i, R_j]\big) + V(U). \tag{34}$$

This is precisely the energy density functional used to define the mass of a static Skyrmion (Taiji knot).

3.4 Maxwell Hamiltonian Density

The electromagnetic Lagrangian density introduced in Eq. (27) has the standard Maxwell form

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4g_{\text{EM}}^2} F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

We work in Minkowski spacetime with metric $\eta_{\mu\nu}$ as in the Skyrme sector. It is convenient to decompose the field strength into electric and magnetic components,

$$E_i \equiv F_{0i}, \qquad B_i \equiv -\frac{1}{2}\epsilon_{ijk}F_{jk}, \tag{35}$$

with ϵ_{ijk} the Levi–Civita symbol in three spatial dimensions. In terms of these variables one finds

$$\mathcal{L}_{\text{EM}} = \frac{1}{2g_{\text{EM}}^2}\left(E_i E_i - B_i B_i\right), \tag{36}$$

where repeated spatial indices are summed over $i = 1, 2, 3$.

The canonical momentum conjugate to the spatial components A_i is

$$\Pi^i(x) \equiv \frac{\partial \mathcal{L}_{\text{EM}}}{\partial(\partial_0 A_i)} = \frac{1}{g_{\text{EM}}^2} E^i(x), \tag{37}$$

while the momentum conjugate to A_0 vanishes,

$$\Pi^0(x) = 0, \tag{38}$$

reflecting the usual primary constraint associated with $U(1)$ gauge invariance.

The electromagnetic Hamiltonian density is obtained in the usual way,

$$\begin{aligned}\mathcal{H}_{\text{EM}} &= \Pi^i \partial_0 A_i - \mathcal{L}_{\text{EM}} \\ &= \frac{g_{\text{EM}}^2}{2}\,\Pi^i\Pi^i + \frac{1}{2g_{\text{EM}}^2}\,B_i B_i + A_0\,\partial_i \Pi^i.\end{aligned} \tag{39}$$

Here the A_0 term enforces Gauss's law $\partial_i \Pi^i = 0$ in the absence of sources. The physical electromagnetic energy density is therefore

$$\mathcal{H}_{\text{EM}}^{\text{phys}} = \frac{g_{\text{EM}}^2}{2}\,\Pi^i\Pi^i + \frac{1}{2g_{\text{EM}}^2}\,B_i B_i = \frac{1}{2g_{\text{EM}}^2}\left(E_i E_i + B_i B_i\right), \tag{40}$$

which is the standard expression for the energy density of the free electromagnetic field.

Including the electromagnetic sector, the full Taiji–Skyrme Hamiltonian density becomes

$$\mathcal{H}_{\text{total}} = \mathcal{H}_{\text{Skyrme}}[U] + \mathcal{H}_{\text{EM}}[A_i, \Pi^i], \tag{41}$$

where $\mathcal{H}_{\text{Skyrme}}[U]$ is given by Eq.24 and $\mathcal{H}_{\text{EM}}$ by Eq. (39).

4 Symmetric Energy–Momentum Tensor

4.1 General Definition

In curved spacetime with metric $g_{\mu\nu}$, the symmetric energy–momentum tensor is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\,\frac{\delta}{\delta g^{\mu\nu}}\Big(\sqrt{-g}\,\mathcal{L}\Big). \tag{42}$$

In flat Minkowski space with $g_{\mu\nu} = \eta_{\mu\nu}$ and $\sqrt{-g} = 1$, and assuming that the metric appears only through index contractions, this reduces to

$$T_{\mu\nu} = 2\,\frac{\partial \mathcal{L}}{\partial \eta^{\mu\nu}} - \eta_{\mu\nu}\,\mathcal{L}. \tag{43}$$

We apply (43) term by term to $\mathcal{L}_2$, $\mathcal{L}_4$, and $V(U)$.

4.2 Quadratic Contribution $T^{(2)}_{\mu\nu}$

From

$$\mathcal{L}_2 = \frac{f^2}{16}\,\eta^{\alpha\beta}\,\mathrm{Tr}\big(R_\alpha R_\beta\big), \tag{44}$$

we obtain

$$\frac{\partial \mathcal{L}_2}{\partial \eta^{\mu\nu}} = \frac{f^2}{16}\,\mathrm{Tr}\big(R_\mu R_\nu\big). \tag{45}$$

Hence

$$T^{(2)}_{\mu\nu} = 2\,\frac{\partial \mathcal{L}_2}{\partial \eta^{\mu\nu}} - \eta_{\mu\nu}\,\mathcal{L}_2 = \frac{f^2}{8}\,\mathrm{Tr}\big(R_\mu R_\nu\big) - \eta_{\mu\nu}\,\frac{f^2}{16}\,\mathrm{Tr}\big(R_\alpha R^\alpha\big). \tag{46}$$

4.3 Skyrme Contribution $T^{(4)}_{\mu\nu}$

The quartic term can be written as

$$\mathcal{L}_4 = \frac{1}{32e^2}\,\eta^{\alpha\gamma}\eta^{\beta\delta}\,\mathrm{Tr}\big([R_\alpha, R_\beta]\,[R_\gamma, R_\delta]\big). \tag{47}$$

Varying with respect to $\eta^{\mu\nu}$ yields the standard compact form

$$\frac{\partial \mathcal{L}_4}{\partial \eta^{\mu\nu}} = \frac{1}{16e^2}\,\mathrm{Tr}\big([R_\mu, R_\alpha]\,[R_\nu, R^\alpha]\big). \tag{48}$$

Therefore

$$T^{(4)}_{\mu\nu} = 2\,\frac{\partial \mathcal{L}_4}{\partial \eta^{\mu\nu}} - \eta_{\mu\nu}\,\mathcal{L}_4 = \frac{1}{8e^2}\,\mathrm{Tr}\big([R_\mu, R_\alpha][R_\nu, R^\alpha]\big) - \eta_{\mu\nu}\,\frac{1}{32e^2}\,\mathrm{Tr}\big([R_\alpha, R_\beta][R^\alpha, R^\beta]\big). \tag{49}$$

4.4 Potential Contribution $T^{(V)}_{\mu\nu}$

The potential $V(U)$ does not depend explicitly on the metric, so

$$\frac{\partial V}{\partial \eta^{\mu\nu}} = 0, \tag{50}$$

and

$$T^{(V)}_{\mu\nu} = -\,\eta_{\mu\nu}\,V(U). \tag{51}$$

4.5 Total Energy–Momentum Tensor

Summing the three pieces,

$$T_{\mu\nu} = T^{(2)}_{\mu\nu} + T^{(4)}_{\mu\nu} + T^{(V)}_{\mu\nu}, \tag{52}$$

we obtain

$$\begin{aligned} T_{\mu\nu} = {} & \frac{f^2}{8}\,\mathrm{Tr}\big(R_\mu R_\nu\big) - \eta_{\mu\nu}\,\frac{f^2}{16}\,\mathrm{Tr}\big(R_\alpha R^\alpha\big) \\ & + \frac{1}{8e^2}\,\mathrm{Tr}\big([R_\mu, R_\alpha][R_\nu, R^\alpha]\big) - \eta_{\mu\nu}\,\frac{1}{32e^2}\,\mathrm{Tr}\big([R_\alpha, R_\beta][R^\alpha, R^\beta]\big) \\ & - \eta_{\mu\nu}\,V(U). \end{aligned} \tag{53}$$

This is the symmetric Skyrme–Taiji energy–momentum tensor in flat spacetime.

4.6 Electromagnetic and Coherence–Connection Contributions

The electromagnetic sector is described by the Maxwell Lagrangian

$$\mathcal{L}_{\mathrm{EM}} = -\,\frac{1}{4g^2_{\mathrm{EM}}}\,F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{54}$$

while the coherence spinor couples to the internal connection through Eq. (19). In a general curved background with metric $g_{\mu\nu}$, the total stress–energy tensor is defined by

$$T^{\mathrm{tot}}_{\mu\nu} = -\,\frac{2}{\sqrt{-g}}\,\frac{\delta}{\delta g^{\mu\nu}} \int d^4x\sqrt{-g}\,\mathcal{L}_{\mathrm{total}}, \tag{55}$$

and can be decomposed as

$$T^{\mathrm{tot}}_{\mu\nu} = T^{(\mathrm{Sk})}_{\mu\nu} + T^{(\hat{Z},A)}_{\mu\nu} + T^{(\mathrm{EM})}_{\mu\nu}, \tag{56}$$

where $T^{(\mathrm{Sk})}_{\mu\nu}$ is the Skyrme contribution already derived in this document.

For the Maxwell field, variation of (54) yields the standard expression

$$T^{(\mathrm{EM})}_{\mu\nu} = \frac{1}{g^2_{\mathrm{EM}}}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{57}$$

This is the usual energy–momentum tensor of the free electromagnetic field.

For the coherence spinor and its covariant coupling to A_μ, the contribution follows from

$$\mathcal{L}_{\hat{Z},A} = \kappa\, g^{\alpha\beta}(D_\alpha\hat{Z})^\dagger D_\beta\hat{Z} - V_{\hat{Z}}(\hat{Z}), \tag{58}$$

and takes the symmetric form

$$T^{(\hat{Z},A)}_{\mu\nu} = \kappa\Big[(D_\mu\hat{Z})^\dagger D_\nu\hat{Z} + (D_\nu\hat{Z})^\dagger D_\mu\hat{Z}\Big] - g_{\mu\nu}\,\mathcal{L}_{\hat{Z},A}. \tag{59}$$

In flat spacetime with $g_{\mu\nu} = \eta_{\mu\nu}$ these expressions reduce to the familiar Minkowski forms; the total energy density appearing in the static limit is then

$$T^{\mathrm{tot}}_{00} = T^{(\mathrm{Sk})}_{00} + T^{(\hat{Z},A)}_{00} + T^{(\mathrm{EM})}_{00}, \tag{60}$$

with the Skyrme part as in the preceding subsections and the electromagnetic parts given by Eqs. (57) and (59).

4.7 Energy Density and Static Limit

The energy density is the 00 component

$$\mathcal{E}(t,\mathbf{x}) \equiv T_{00}(t,\mathbf{x}). \tag{61}$$

Using $\eta_{00} = +1$ and $R^0 = R_0$, we find

$$\begin{aligned} T_{00} = {} & \frac{f^2}{8}\,\mathrm{Tr}\big(R_0 R_0\big) - \eta_{00}\,\frac{f^2}{16}\,\mathrm{Tr}\big(R_\alpha R^\alpha\big) \\ & + \frac{1}{8e^2}\,\mathrm{Tr}\big([R_0,R_\alpha][R_0,R^\alpha]\big) - \eta_{00}\,\frac{1}{32e^2}\,\mathrm{Tr}\big([R_\alpha,R_\beta][R^\alpha,R^\beta]\big) \\ & - \eta_{00}\,V(U). \end{aligned} \tag{62}$$

In the static limit ($R_0 = 0$), this simplifies to

$$T_{00}^{(\mathrm{static})} = \frac{f^2}{16}\,\mathrm{Tr}(R_i R_i) + \frac{1}{32e^2}\,\mathrm{Tr}\big([R_i,R_j][R_i,R_j]\big) + V(U), \tag{63}$$

which coincides with the static Hamiltonian density (34). The mass of a static Skyrmion (Taiji knot) is then

$$M = \int d^3x\; T_{00}^{(\mathrm{static})}. \tag{64}$$

5 Static Hedgehog Ansatz (Reference)

For completeness, we recall the standard hedgehog ansatz used in the Taiji–Skyrme context. Spherically symmetric static configurations are written as

$$U(\mathbf{x}) = \cos F(r)\,\mathbf{1} + i\,(\hat{\mathbf{x}}\cdot\boldsymbol{\sigma})\,\sin F(r), \qquad r = |\mathbf{x}|, \tag{65}$$

where $F(r)$ is a real profile function satisfying

$$F(0) = \pi, \qquad \lim_{r\to\infty} F(r) = 0. \tag{66}$$

Substituting this ansatz into the static Hamiltonian density (34), one obtains a radial energy functional whose Euler–Lagrange equation gives the ordinary differential equation for $F(r)$. The resulting solution represents a topologically nontrivial configuration with baryon number (topological charge) $Q = 1$.

6 Summary

In this document we have formulated the Skyrme–type Taiji model in a self–contained manner. The Taiji spinor doublet is mapped to an SU(2) field $U(x)$, and the dynamics is governed by the Skyrme Lagrangian with a mass potential. We derived:

- the full Lagrangian density in terms of the right–invariant current R_μ,
- the Hamiltonian density in Minkowski spacetime,
- the symmetric energy–momentum tensor $T_{\mu\nu}$,
- and the static energy density for Skyrmion–like Taiji knots.

This framework provides the mathematical foundation for extending the Taiji theory into a topological matter sector and for coupling Skyrmionic Taiji knots to gravity in subsequent work.