Quantum Coherence in the Skyrme–Taiji Framework

Eduardo Santibañez

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Abstract

This document develops a complete quantum–mechanical framework for coherence in the Skyrme–Taiji theory. Starting from the unified Taiji spinor and its associated SU(2) field, we construct the canonical phase space, current algebra, and Finkelstein–Rubinstein quantization of topological sectors, providing the operator foundation on which quantum coherence must be defined.

On this basis we introduce a renormalised composite coherence operator $\hat{C}_R(x)$ built from Skyrme currents and analyse its algebra, dimensional properties, and effective field–theory status. The classical coherence field and its dimensionless order parameter $\psi(x)$ arise as expectation values and normalised scalars, linking the quantum operator to the coherence dictionary used in the vacuum, cosmological, and phenomenological sectors of the Taiji–Skyrme program.

We then study linearised quantum fluctuations around vacuum and solitonic backgrounds, derive the quadratic fluctuation operator and its normal modes, and identify the corresponding coherence fluctuation spectrum and stability conditions. This establishes coherence as a bona fide quantum degree of freedom with its own excitations and uncertainty relations, rather than as a purely phenomenological construct.

Finally, we formulate the path–integral description of coherence, define generating functionals and coherence kernels, and discuss decoherence and influence functionals for coarse grained coherence variables. We analyse how topological charge, FR sectors, and coherence correlations shape entanglement patterns between spatial regions and solitonic degrees of freedom. Taken together, these results close the quantum layer of the Skyrme–Taiji coherence hierarchy and provide the operator-level foundations required for subsequent applications across the Taiji–Skyrme coherence framework.

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1 Introduction

1.1 Purpose of the Coherence Framework

The purpose of this document is to construct a complete quantum–mechanical formalism for coherence within the Skyrme–Taiji field theory. The Taiji program begins from the physical premise that the fundamental degrees of freedom of nature are represented not by individual particles, but by a unified order field whose excitations exhibit both geometric and topological structure. In the Skyrme–Taiji framework, the elementary excitations of this field are topological solitons whose stability is guaranteed by the nontrivial homotopy of the vacuum manifold.

From this viewpoint, coherence is not an auxiliary or emergent quantity, but a central structural observable that captures how ordered, aligned and correlated the field is across scales. A quantum theory of coherence must therefore be built at the same level of rigor as the usual quantum field observables: it must start from a precise specification of the field content and symmetries, proceed through a consistent quantization procedure, and culminate in well–defined operators, spectra and correlation functions.

Concretely, a quantum coherence framework must provide:

1. a precise specification of the field content and vacuum manifold;
2. a complete canonical (and path–integral) quantization scheme;
3. a rigorous definition of local and global quantum coherence operators;
4. the derivation of coherence spectra and correlators around relevant classical backgrounds (e.g. solitons);
5. a formulation of decoherence and entanglement laws compatible with the topological sector structure.

The present document develops these foundations and prepares the ground for quantum–level predictions of Taiji dynamics.

1.2 From Excitation–Potential Field Duality to Unified Spinor Field

The original Taiji interpretation distinguishes two conceptual ingredients: an Excitation–like ordering field and a Potential Field–like ground. Classically, these may be treated as separate entities, one enforcing structure and one providing a continuous medium of possibilities. In the Skyrme–Taiji construction it is advantageous to assemble them into a single dynamical object: a complex two–component spinor

$$Z(x) = \begin{pmatrix} \Psi(x) \\ \Phi(x) \end{pmatrix} \in \mathbb{C}^2, \tag{1}$$

transforming in the fundamental representation of $SU(2)$. This unification has several immediate consequences:

- It provides a compact vacuum manifold (a three–sphere) on which the field takes its vacuum values.
- It allows the Excitation–like and Potential Field–like aspects to be viewed as different components of a single coherent degree of freedom.
- It naturally incorporates the internal $SU(2)$ structure required for topological solitons of Skyrme type.

In this sense, the Excitation–Potential Field duality is not discarded but re–encoded as the internal spinor structure of $Z(x)$. The quantum coherence properties we want to study will ultimately be defined in terms of this unified spinor and its associated $SU(2)$ orientation.

1.3 The Need for Topological Quantization

The vacuum manifold of the Skyrme–Taiji model, as we will see explicitly below, is topologically a three–sphere S^3. Static finite–energy configurations are therefore classified by the homotopy group

$$\pi_3(S^3) \cong \mathbb{Z}, \tag{2}$$

and each integer $Q \in \mathbb{Z}$ labels a distinct topological sector. Configurations with different Q cannot be continuously deformed into one another without leaving the space of finite–energy fields.

A quantum theory of coherence must respect this sector structure. In particular:

- the Hilbert space must decompose into subspaces labelled by the topological charge Q;
- the coherence operators must act consistently within or between these sectors;
- the quantization procedure must correctly encode the nontrivial topology of configuration space.

The last point is crucial: topology can modify quantum statistics and the behavior of wavefunctionals under large transformations. In the Skyrme–Taiji setting, this will lead to a topological quantization condition of Finkelstein–Rubinstein type, ensuring that solitons with odd topological charge behave as fermions.

1.4 Relation to Taiji Physical Principles

Within the broader Taiji interpretation, coherence measures the degree of structural alignment across scales: how consistently the underlying order field organizes itself from local to global configurations. In the Skyrme–Taiji model, this concept is sharpened: coherence becomes an operator–valued quantity constructed directly from the spinor field $Z(x)$ and its $SU(2)$ orientation $U(x)$.

Thus, coherence acquires a fully quantum meaning: it is represented by local and global operators, it admits a spectral decomposition, it obeys uncertainty relations, and it evolves according to the underlying Hamiltonian or path–integral dynamics. The aim of this document is to make these statements explicit and computable, thereby linking the qualitative Taiji notion of coherence to a concrete quantum field theoretic formalism.

1.5 Coherence Hierarchy and Dictionary Reference.

In the Taiji–Skyrme framework, the renormalised composite operator $\hat{C}_R(x)$ constructed here is the fundamental quantum coherence observable. Its expectation value in Excitation-selected quantum states defines the classical coherence density $C_{\mathrm{cl}}(x)$, and the dimensionless coherence order parameter used throughout the effective theory is obtained by normalising C_{cl} with respect to a fixed reference scale $C_\star$. The full hierarchy of definitions is summarised in the standalone document *Coherence Dictionary for the Taiji–Skyrme Framework.*

2 Field Foundations

2.1 The Taiji Spinor Field

We introduce the unified Taiji spinor field

$$Z(x) = \begin{pmatrix} \Psi(x) \\ \Phi(x) \end{pmatrix}, \qquad \Psi(x), \Phi(x) \in \mathbb{C}. \tag{3}$$

The squared magnitude

$$|Z(x)|^2 = |\Psi(x)|^2 + |\Phi(x)|^2 \tag{4}$$

plays the role of a local coherence amplitude: it measures how strongly the Excitation–like and Potential Field–like components are present at the point x. Configurations where $|Z|$ is close to its vacuum value will correspond to highly ordered regions of the field, while deviations encode local excitations and fluctuations.

In this language, the basic dynamical degree of freedom is not a collection of separate particle fields, but a single spinor encoding the interplay between order (Excitation, Φ) and potentiality (Potential Field, Ψ). The subsequent constructions of the vacuum manifold, the $SU(2)$ group element and the Skyrme–Taiji Lagrangian all follow from this choice.

2.2 Coherence Internal Space and Local Phase Redundancy

In the previous subsection we introduced the Taiji spinor

$$Z(x) = \begin{pmatrix} \Psi(x) \\ \Phi(x) \end{pmatrix}, \qquad x^\mu = (t, \mathbf{x}),$$

with classical coherence amplitude

$$C(x) \equiv |Z(x)|^2 = Z^\dagger(x) Z(x) = |\Psi(x)|^2 + |\Phi(x)|^2 \geq 0.$$

Whenever $C(x) > 0$ we defined the unit–norm spinor

$$\hat{Z}(x) \equiv \frac{Z(x)}{\sqrt{C(x)}}, \qquad \hat{Z}^\dagger(x)\, \hat{Z}(x) = 1,$$

and used $\hat{Z}$ to construct the $SU(2)$ field $U(x)$ that furnishes the classical Skyrme dynamics.

At this level, $\hat{Z}$ is not unique. Multiplying $\hat{Z}$ at each spacetime point by a phase factor leaves both its norm and the associated $SU(2)$ element unchanged. Explicitly, for any real function $\lambda(x)$ we have the local phase transformation

$$\hat{Z}(x) \;\longrightarrow\; \hat{Z}'(x) = e^{i\lambda(x)}\, \hat{Z}(x), \tag{5}$$

under which

$$\hat{Z}'^\dagger \hat{Z}' = \hat{Z}^\dagger \hat{Z} = 1.$$

The map from $\hat{Z}$ to U is insensitive to this phase: different representatives $\hat{Z}$ and $e^{i\lambda(x)}\hat{Z}$ correspond to the same $SU(2)$ field $U(x)$. Thus the physically relevant internal coherence state is not the spinor $\hat{Z}$ itself, but its equivalence class under local phase rotations.

This identifies the internal coherence space with the complex projective line

$$\mathrm{CP}^1 \simeq S^2,$$

where each point corresponds to a ray $[\hat{Z}]$ rather than to a specific spinor representative. In particular, the classical alignment variables can be expressed as

$$n^a(x) \equiv \hat{Z}^\dagger(x)\, \sigma^a\, \hat{Z}(x), \qquad a = 1, 2, 3,$$

with

$$n^a n^a = 1,$$

so that $n^a(x)$ parametrises a point on the internal two–sphere S^2 at each spacetime point.

The transformation (5) is a *redundancy* in the description of the same physical coherence configuration. Two spinors related by a local phase represent the same internal direction $n^a(x)$ and the same $SU(2)$ field $U(x)$. We will therefore treat (5) as a local $U(1)$ gauge freedom in the coherence sector.

Coherence Connection as an Internal Gauge Field

Once the local phase redundancy has been identified, it is natural to introduce a connection that tracks how the coherence state twists in internal space as one moves through spacetime. A convenient choice is the one–form

$$A_\mu(x) \equiv -\,i\,\hat{Z}^\dagger(x)\,\partial_\mu \hat{Z}(x), \qquad \mu = 0,1,2,3, \tag{6}$$

which is real by construction. Under a local phase transformation (5),

$$\hat{Z}(x) \to \hat{Z}'(x) = e^{i\lambda(x)}\,\hat{Z}(x), \tag{7}$$

$$\partial_\mu \hat{Z}(x) \to \partial_\mu \hat{Z}'(x) = e^{i\lambda(x)}\big(\partial_\mu \hat{Z}(x) + i(\partial_\mu \lambda(x))\hat{Z}(x)\big), \tag{8}$$

so that

$$\begin{aligned} A'_\mu(x) &= -\,i\,\hat{Z}'^\dagger(x)\,\partial_\mu \hat{Z}'(x) \\ &= -\,i\,\hat{Z}^\dagger(x)\,\big(\partial_\mu \hat{Z}(x) + i(\partial_\mu\lambda(x))\hat{Z}(x)\big) \\ &= -\,i\,\hat{Z}^\dagger(x)\,\partial_\mu \hat{Z}(x) + \partial_\mu \lambda(x) \\ &= A_\mu(x) + \partial_\mu \lambda(x). \end{aligned} \tag{9}$$

Thus the coherence connection A_μ transforms as

$$A_\mu \to A_\mu + \partial_\mu \lambda,$$

which is precisely the $U(1)$ gauge transformation law satisfied by an electromagnetic vector potential.

We emphasise that at this stage A_μ is defined as a *composite* object built from $\hat{Z}$; no independent dynamics has yet been assigned to it. The identification (6)–(9) shows that the internal phase redundancy of the Taiji spinor naturally supports a $U(1)$ connection. In later sections this observation will be used to define a Maxwell-type dynamical sector in which A_μ becomes the electromagnetic field associated with coherence, and its curvature

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

will play the role of the classical electromagnetic field strength.

2.3 Vacuum Manifold and Symmetry Breaking

To obtain a nontrivial vacuum structure, we adopt a symmetry–breaking potential for Z of the form

$$V(Z) = \frac{\lambda}{4}\left(|Z|^2 - v^2\right)^2, \qquad \lambda > 0,\ v > 0. \tag{10}$$

The minima of this potential satisfy

$$|Z| = v, \tag{11}$$

which defines a three–sphere

$$S^3 = \{Z \in \mathbb{C}^2 \mid |Z| = v\} \subset \mathbb{C}^2. \tag{12}$$

Thus the vacuum manifold is

$$\mathcal{M}_{\text{vac}} \simeq S^3. \tag{13}$$

The key topological property is the nontrivial homotopy group

$$\pi_3(S^3) \cong \mathbb{Z}, \tag{14}$$

which ensures the existence of stable topological solitons (Skyrmions) classified by an integer winding number Q. These solitons will represent the basic matter–like excitations in the Skyrme–Taiji framework, and the quantum coherence formalism developed later will apply in particular to their fluctuations and interactions.

2.4 Embedding into $SU(2)$

It is convenient to factor out the overall magnitude of Z and work with a normalized spinor

$$\hat{Z}(x) = \frac{Z(x)}{|Z(x)|}, \qquad |\hat{Z}(x)| = 1 \tag{15}$$

whenever $Z(x) \neq 0$. This normalized spinor takes values on the unit three–sphere S^3 and can be mapped to an $SU(2)$ group element. Writing $\hat{Z}$ in components as

$$\hat{Z}(x) = \begin{pmatrix} \hat{Z}_0(x) \\ \hat{Z}_1(x) \end{pmatrix}, \tag{16}$$

we define

$$U(x) = \hat{Z}_0(x)\,\mathbf{1} + i\,\hat{Z}_i(x)\,\sigma_i \in SU(2), \tag{17}$$

where σ_i are the Pauli matrices and the components $\hat{Z}_i$ are chosen such that $U(x)$ is unitary with unit determinant. This identification realizes the vacuum manifold as the group manifold of $SU(2)$ and makes manifest the internal $SU(2)$ structure of the field.

In practice, we will treat $U(x)$ as the primary field variable in the Skyrme–Taiji Lagrangian. The underlying spinor $Z(x)$ remains important conceptually, as it carries the Excitation–Potential Field interpretation, but many of the geometric and topological constructions are more naturally expressed in terms of U.

2.5 The Skyrme–Taiji Lagrangian

The dynamical content of the model is captured by an $SU(2)$ Skyrme–type Lagrangian density for $U(x)$, supplemented by a potential inherited from the symmetry–breaking structure of Z:

$$\mathcal{L} = \frac{f^2}{4}\,\mathrm{Tr}\Big(\partial_\mu U^\dagger \partial^\mu U\Big) + \frac{1}{32e^2}\,\mathrm{Tr}\Big([U^\dagger\partial_\mu U, U^\dagger \partial_\nu U]^2\Big) - V(U). \tag{18}$$

Here f and e are positive constants setting the overall energy and length scales of the model, and $V(U)$ is chosen such that its minima reproduce the vacuum manifold previously described (e.g. by expressing V in terms of the underlying Z or appropriate invariants of U).

The first term is the usual nonlinear sigma model kinetic term: it measures how fast the field configuration moves on the group manifold. The second term is the Skyrme term: it stabilizes finite–energy solitons by penalizing large gradients and preventing collapse. Together, these terms yield a field theory admitting stable topological solitons, whose quantum fluctuations and coherence properties will be the focus of the subsequent sections.

In summary, the field foundations of the quantum coherence framework consist of:

- a unified Taiji spinor $Z(x)$ encoding Excitation (Φ) and Potential Field (Ψ) components,
- a vacuum manifold $\mathcal{M}_{\mathrm{vac}} \simeq S^3$ with nontrivial topology,
- an $SU(2)$ group element $U(x)$ obtained from the normalized spinor,
- a Skyrme–type Lagrangian $\mathcal{L}[U]$ supporting stable solitons.

The next step is to quantize this system and to define quantum coherence operators in a way that respects both the group structure and the topological sector decomposition.

3 Canonical Quantization Framework

3.1 Overview and Physical Motivation

To develop a quantum theory of coherence for the Skyrme–Taiji field, we must quantize the $SU(2)$ group–valued field $U(x)$ and its associated excitations. Because the field configurations of finite energy must approach a constant value at spatial infinity, the physical configuration space splits into topological sectors labelled by the winding number $Q \in \mathbb{Z}$. A consistent quantization procedure must therefore respect:

1. the nonlinear group structure of $SU(2)$,
2. the nontrivial topology of configuration space,
3. the existence of stable solitons (topological knots),
4. the Excitation–Potential Field interpretation encoded in the spinor $\hat{Z}(x)$.

Canonical quantization begins by identifying the classical phase space, constructing Poisson brackets, and promoting these to operator commutation relations. Because $U(x) \in SU(2)$ lives on a curved group manifold rather than a vector space, the correct phase space structure uses left– and right–invariant currents, which form the Maurer–Cartan frames of the group.

This section lays out the canonical formalism needed for quantum coherence operators and prepares the ground for the Finkelstein–Rubinstein quantization condition introduced at the end.

3.2 Maurer–Cartan Currents and Phase Space Variables

The natural phase–space variables for an $SU(2)$ group–valued field are the Maurer–Cartan currents:

$$L_\mu = U^\dagger \partial_\mu U, \tag{19}$$

$$R_\mu = \partial_\mu U\, U^\dagger, \tag{20}$$

which satisfy the well–known Maurer–Cartan identity

$$\partial_\mu L_\nu - \partial_\nu L_\mu + [L_\mu, L_\nu] = 0. \tag{21}$$

Since the Skyrme–Taiji Lagrangian depends on derivatives of U, these currents organize the dynamical content in a form that respects the underlying group geometry.

The canonical momenta conjugate to U may be expressed in terms of the time component of the left–invariant current,

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 U)} U^\dagger. \tag{22}$$

These momenta take values in the Lie algebra $\mathfrak{su}(2)$, and together (U, Π) form the basic phase–space coordinates.

3.3 Classical Poisson Brackets

The Poisson bracket structure for $SU(2)$–valued fields follows from their Lie–algebraic nature. The currents satisfy

$$\{L_i(x), L_j(y)\} = \epsilon_{ijk} L_k(x) \delta(x-y), \tag{23}$$

$$\{R_i(x), R_j(y)\} = -\epsilon_{ijk} R_k(x) \delta(x-y), \tag{24}$$

$$\{L_i(x), R_j(y)\} = 0, \tag{25}$$

where $L_i = \frac{1}{2}\mathrm{Tr}(\sigma_i L)$, etc. These relations encode the local $SU(2)$ symmetry and ensure that the group structure is preserved under time evolution.

The Poisson brackets between U and its conjugate momenta Π are

$$\{U_{ab}(x), \Pi_{cd}(y)\} = \delta_{ad}\delta_{bc}\delta(x-y), \tag{26}$$

with all other brackets vanishing or obtained by Hermitian conjugation.

3.4 Canonical Commutation Relations

Canonical quantization proceeds by promoting phase–space variables to operators:

$$U(x) \to \hat{U}(x), \qquad \Pi(x) \to \hat{\Pi}(x),$$

and replacing Poisson brackets with commutators:

$$\{A, B\} \;\longrightarrow\; \frac{1}{i\hbar}[\hat{A}, \hat{B}].$$

The basic equal–time commutation relations are

$$[\hat{U}_{ab}(x), \hat{\Pi}_{cd}(y)] = i\hbar\,\delta_{ad}\delta_{bc}\,\delta(x-y), \tag{27}$$

$$[\hat{L}_i(x), \hat{L}_j(y)] = i\hbar\,\epsilon_{ijk}\hat{L}_k(x)\delta(x-y), \tag{28}$$

$$[\hat{R}_i(x), \hat{R}_j(y)] = -i\hbar\,\epsilon_{ijk}\hat{R}_k(x)\delta(x-y), \tag{29}$$

with all mixed commutators vanishing. These relations encode the $SU(2)$ group geometry at the quantum level.

3.5 Wavefunctionals and Configuration Space Sectors

A quantum state of the system is described by a wavefunctional

$$\Psi[U],$$

which assigns a complex amplitude to each configuration $U(x)$. Because the physical configuration space decomposes into topological sectors

$$\mathcal{C} = \bigcup_{Q\in\mathbb{Z}} \mathcal{C}_Q,$$

the Hilbert space must carry the same decomposition:

$$\mathcal{H} = \bigoplus_{Q} \mathcal{H}_Q.$$

Operators that change the topological sector must be carefully defined, and in many contexts (e.g. soliton quantization) one works within a fixed Q–sector. The topology of $\mathcal{C}_Q$ has direct consequences for quantum statistics, as explained in the next subsection.

3.6 Finkelstein–Rubinstein Quantization

The configuration space $\mathcal{C}_Q$ of finite–energy field configurations in a fixed topological sector Q has nontrivial fundamental group:

$$\pi_1(\mathcal{C}_Q) \cong \begin{cases} \mathbb{Z}_2, & Q \text{ odd}, \\ 0, & Q \text{ even}. \end{cases} \tag{30}$$

This means that for odd Q there exist loops in configuration space that cannot be continuously contracted to a point. One such loop corresponds physically to a 2π spatial rotation of the entire field configuration.

A quantum wavefunctional must transform under such a loop according to

$$\Psi[U_{2\pi}] = e^{i\alpha}\,\Psi[U_0], \tag{31}$$

where $U_{2\pi}$ is the configuration obtained after a full rotation. Finkelstein and Rubinstein showed that consistency of the quantum theory requires

$$e^{i\alpha} = (-1)^Q. \tag{32}$$

Therefore:

- If Q is odd, a 2π rotation changes the sign of the wavefunctional: the soliton behaves as a fermion.
- If Q is even, no sign change occurs: the soliton behaves as a boson.

This topological rule fixes the spin–statistics relation for Skyrme–Taiji solitons without introducing additional fields or higher–dimensional terms. In particular, single–knot solitons ($Q = 1$) quantize to spin–$\frac{1}{2}$ fermions, matching the physical behavior expected of matter–like excitations in the Taiji interpretation.

3.7 Summary of Canonical Structure

The canonical quantization framework consists of:

- Maurer–Cartan currents encoding $SU(2)$ geometry,
- Poisson brackets promoting the group structure to phase space,
- operator commutation relations implementing the quantum theory,
- wavefunctionals defined on topological sectors $\mathcal{C}_Q$,
- the Finkelstein–Rubinstein condition ensuring correct quantum statistics for solitons.

This framework prepares the ground for the definition and analysis of quantum coherence operators, which we now construct.

4 The Quantum Coherence Operator

4.1 Motivation and Conceptual Role

In the Taiji–Skyrme framework, coherence represents the intrinsic degree of internal organization of the $SU(2)$ field. It encodes how uniformly the underlying order is distributed across space and time and identifies the regions in which the field maintains its structure most robustly. Because this organization contributes directly to the local energy density, coherence plays a structural role in the dynamics of the theory: it influences the stress–energy tensor that sources curvature, but it does not act as an independent dynamical force.

A quantum formulation of coherence therefore requires a precise operator that captures this internal organization at the quantum level. The construction proceeds by establishing the following elements in a systematic manner:

1. **Classical invariant:** Define a coherence density from the left–invariant $SU(2)$ currents, ensuring that the quantity is frame–independent and geometrically meaningful.

2. **Operator promotion:** Lift this classical density to a quantum operator compatible with the canonical current algebra derived earlier, so that coherence can act on wavefunctionals $\Psi[U]$.

3. **Regularization:** Introduce a consistent short–distance regulator to control the divergences inherent in composite operators constructed from products of currents evaluated at the same point.

4. **Renormalization:** Remove the regulator in a controlled way by defining a finite, renormalized operator $\mathcal{C}_R(x)$ whose expectation values and correlation functions remain well defined.

The resulting operator provides the quantum–mechanical foundation for all subsequent analyses of coherence fluctuations, spectra, kernels, and entanglement.

4.2 Classical Coherence Density

A natural classical measure of local coherence is provided by the left–current $L_\mu = U^\dagger \partial_\mu U$. In particular, the spatial components L_i (with $i = 1, 2, 3$) capture how the field U twists and rotates across space. The simplest scalar density built from L_i is

$$\mathcal{C}(x) = -\frac{f^2}{4} \operatorname{Tr}(L_i L_i)\,, \qquad i = 1, 2, 3. \tag{33}$$

Because the trace is invariant under local and global $SU(2)$ transformations, this quantity measures the strength of spatial alignment (or misalignment) in a gauge–independent way.

In classical soliton solutions, regions of strong coherence correspond to where $U(x)$ changes slowly, while regions of decreased coherence mark zones of distortion or excitation. The integral of $\mathcal{C}(x)$ over space provides a global measure of the ordered structure of the configuration.

4.3 Operator Construction and Normal Ordering

Upon quantization, $U(x)$ and $L_i(x)$ become operator–valued distributions. Products like $L_i L_i$ evaluated at the same spacetime point develop short–distance singularities, reflecting the usual ultraviolet divergences of local composite operators.

To construct a well–defined quantum coherence operator, one introduces a point–splitting regulator:

$$\hat{\mathcal{C}}_\epsilon(x) = -\frac{f^2}{4} \operatorname{Tr}\Big[\hat{L}_i(x+\epsilon)\, \hat{L}_i(x-\epsilon)\Big]\,, \tag{34}$$

and then defines the renormalized operator by subtracting the divergent piece:

$$\hat{\mathcal{C}}_R(x) = \lim_{\epsilon \to 0^+} \Big[\hat{\mathcal{C}}_\epsilon(x) - Z_\epsilon\Big]\,, \tag{35}$$

where Z_ϵ encodes the leading short–distance singularity. This defines a finite, well–behaved operator whose expectation values in coherent states, soliton states, or vacuum states correspond to physically meaningful measures of quantum coherence.

4.4 The Coherence Kernel and Operator Algebra

The renormalized operator $\hat{\mathcal{C}}_R(x)$ obeys a nontrivial operator algebra. In particular, its commutator at two distinct spatial points defines the *coherence kernel*:

$$[\,\hat{\mathcal{C}}_R(x),\, \hat{\mathcal{C}}_R(y)\,] = i\hbar\, \mathcal{F}(x, y). \tag{36}$$

The kernel $\mathcal{F}(x,y)$ is a distribution determined by the commutation relations of the underlying currents $\hat{L}_i$. Explicitly, it is built from combinations of δ–functions, derivatives thereof, and group–theoretic coefficients. Although we will not enumerate every term here, we emphasize that the kernel $\mathcal{F}(x,y)$:

- measures how coherence at x influences coherence at y,
- encodes the quantum correlations and causal structure of the field,
- is the generator of coherence transformations,
- determines the coherence uncertainty relation.

Because $\mathcal{C}_R$ is a composite operator built from the left currents, its algebra is completely fixed by the canonical commutation relations of Section 3. This ensures that the coherence operator is not an ad hoc observable, but a consistently quantized field operator.

4.5 UV Cutoff and Effective Field Theory Status

The Taiji–Skyrme model in $(3+1)$ dimensions is a non–renormalizable field theory in the usual perturbative sense. Its Lagrangian contains derivative self–interactions of the $SU(2)$ field that generate an infinite tower of higher–dimension operators when quantum corrections are iterated. As a result, the theory must be interpreted as an effective field theory (EFT), valid only below a finite ultraviolet cutoff scale.

We denote this cutoff by Λ_{T}. It is naturally set by the intrinsic Taiji–Skyrme parameters e and f through a relation of the schematic form

$$\Lambda_{\mathrm{T}} \sim c_\Lambda \, ef, \tag{37}$$

where c_Λ is a dimensionless constant of order unity that depends on the precise definition of the cutoff and matching scheme. For momenta $k \ll \Lambda_{\mathrm{T}}$, the truncated Taiji–Skyrme Lagrangian provides a controlled and predictive description of the dynamics.

The coherence density is a composite operator built from the $SU(2)$ currents, and its naive definition involves products of fields at the same spacetime point. To render this quantity finite within the EFT regime, we introduce a short–distance regulator tied to the cutoff Λ_{T} and define the renormalized operator $\mathcal{C}_R(x)$ by subtracting the divergent, cutoff–dependent pieces. All expectation values and correlation functions of $\mathcal{C}_R(x)$ are to be understood as defined for scales

$$|x-y|^{-1} \ll \Lambda_{\mathrm{T}}, \tag{38}$$

where the effective description is valid.

In this Wilsonian sense, the renormalization of $\mathcal{C}_R(x)$ does not aim at constructing a ultraviolet–complete theory, but at organizing the low–energy behaviour in a consistent way. The effect of modes with $k \gtrsim \Lambda_{\mathrm{T}}$ is absorbed into the cutoff–dependent coefficients of the effective Lagrangian and into the finite normalization of the coherence operator. The quantum coherence framework developed in this work is therefore explicitly understood as a low–energy description, controlled by the hierarchy between the physical scales of interest and the cutoff scale Λ_{T}.

4.6 Coherence Uncertainty Principle

The nonvanishing commutator leads to a coherence uncertainty relation:

$$\Delta\mathcal{C}(x)\,\Delta\mathcal{C}(y) \;\geq\; \frac{\hbar}{2}\,|\,\mathcal{F}(x,y)\,|\,. \tag{39}$$

This relation expresses a fundamental limitation: coherence cannot be made simultaneously sharp at distinct points if $\mathcal{F}(x,y)$ is nonzero. In other words, coherence is not merely a classical alignment property but a quantum observable subject to constraints. This has important implications for:

- the structure of quantum solitons,
- the dynamics of defects and excitations,
- the transport of coherence across space,
- the generation of entanglement between regions.

The uncertainty relation also underpins the quantum corrections to the energy, as regions of high coherence cannot remain perfectly stable under quantization — leading to fluctuations explored in the next section.

4.7 Classical Limit and Coherence Order Parameter

The operator $\hat{C}_R(x)$ defined in Eqs. (22)–(27) is the fundamental coherence observable of the Taiji–Skyrme model. Given an Excitation-selected quantum state $|\Psi\rangle$ describing a semiclassical configuration of the Skyrme field, the classical coherence field is defined as

$$C_{\rm cl}(x) = \langle\Psi|\hat{C}_R(x)|\Psi\rangle. \tag{40}$$

This quantity has engineering dimension of an energy density and is taken as the classical limit of quantum coherence. A fixed reference scale $C_\star$ is introduced to define the dimensionless coherence order parameter used throughout the effective theory:

$$\psi(x) = \frac{C_{\rm cl}(x)}{C_\star}. \tag{41}$$

The field $\psi(x)$ appearing in the effective Lagrangians, vacuum-branch analysis, domain-wall dynamics, galaxy phenomenology, and cosmology is *exactly* this normalised classical expectation value. Any use of spinor parametrisations or energy-density calibrations in other documents is to be interpreted as a representation of the same underlying coherence field $C_{\rm cl}(x)$ rather than as alternative definitions.

This subsection closes the link between the quantum definition of coherence and its classical and phenomenological usage in the broader Taiji–Skyrme framework.

4.8 Summary

In this section we have constructed the renormalised coherence operator $\hat{C}_R(x)$ from the local Skyrme currents using point–splitting and subtraction of short–distance singularities. The resulting operator is finite, transforms trivially under global $SU(2)$ rotations, and retains the engineering dimension of an energy density. Its algebra is fully determined by the canonical commutation relations of the currents, and the associated coherence fluctuations follow from the operator commutators established in this construction.

Section 4.7 introduced the classical limit of this operator through the expectation value $C_{\rm cl}(x) = \langle\Psi|\hat{C}_R(x)|\Psi\rangle$, which defines the classical coherence field in Excitation–selected semiclassical states. Normalisation by a fixed reference scale $C_\star$ yields the dimensionless coherence order parameter $\psi(x) = C_{\rm cl}(x)/C_\star$, which is the quantity that appears in all effective Taiji–Skyrme models, including vacuum-branch structure, domain-wall solutions, galaxy phenomenology, and cosmology. All classical uses of $\psi(x)$ in the broader framework are therefore to be understood as the classical reduction of the quantum operator $\hat{C}_R(x)$ constructed here.

The combination of the renormalised operator definition, its algebraic properties, and the classical mapping ($\hat{C}_R \to C_{\rm cl} \to \psi$) provides the bridge between the quantum formulation and the effective classical dynamics used elsewhere in the theory.

5 Linearized Quantum Fluctuations

5.1 Motivation and Physical Role

To understand quantum coherence in the Skyrme–Taiji framework, one must examine how the field behaves under small quantum fluctuations around a classical background. Such backgrounds may include:

- the vacuum configuration $U = \mathbf{1}$,
- static topological solitons (knots) with charge Q,
- slowly varying collective modes of the soliton manifold.

Linearizing the theory around a chosen background allows us to extract:

1. the spectrum of small excitations,
2. the stability properties of the background,
3. the propagation of coherence fluctuations,
4. quantum corrections to soliton energies,
5. the behavior of entanglement and decoherence modes.

These are the quantum analogues of classical perturbations, and together they define the "quantum coherence spectrum" of the model.

5.2 Expanding the Field about a Background

Let $U_0(x)$ be a classical solution of the Skyrme–Taiji equations of motion. We introduce small fluctuations via:

$$U(x) = U_0(x) \exp\big(i\,\xi^a(x)\sigma_a\big), \tag{42}$$

where $\xi^a(x)$ are real perturbation fields and σ_a are Pauli matrices. To quadratic order in ξ^a, this expansion captures the full linearized dynamics of small deviations from U_0.

The left current becomes:

$$L_\mu = U^\dagger \partial_\mu U = L_\mu^{(0)} + i\big(\partial_\mu \xi^a \sigma_a + [\xi^a \sigma_a, L_\mu^{(0)}]\big) + \mathcal{O}(\xi^2). \tag{43}$$

Here $L_\mu^{(0)}$ is the current associated with the background configuration.

5.3 Quadratic Action and Fluctuation Operator

Inserting the expansion of U into the Skyrme–Taiji Lagrangian and retaining terms up to quadratic order yields the linearized Lagrangian:

$$\mathcal{L}^{(2)} = \frac{1}{2}\,\xi^a(x)\,\mathcal{D}_{ab}\,\xi^b(x), \tag{44}$$

where $\mathcal{D}_{ab}$ is the *fluctuation operator*. Explicitly,

$$\mathcal{D}_{ab} = -f^2 \delta_{ab} \partial_\mu \partial^\mu - 2f^2 \epsilon_{abc}\, L_\mu^{(0)c} \partial^\mu + \mathcal{V}_{ab}[U_0], \tag{45}$$

where $\mathcal{V}_{ab}[U_0]$ is a potential matrix determined by the specific background configuration.

This operator is self–adjoint with respect to the inner product

$$\langle \xi, \zeta \rangle = \int \mathrm{d}^3x\, \xi^a(x) \zeta^a(x), \tag{46}$$

ensuring that its spectrum is real.

5.4 Normal Mode Decomposition

Quantum fluctuations can be decomposed into eigenmodes of the fluctuation operator:

$$\mathcal{D}_{ab}\,\phi_b^{(n)}(x) = \omega_n^2\,\phi_a^{(n)}(x), \tag{47}$$

where ω_n^2 are the squared frequencies of the normal modes.

The perturbation field then expands as:

$$\xi^a(x,t) = \sum_n \left(a_n\,\phi_a^{(n)}(x) e^{-i\omega_n t} + a_n^\dagger\,\phi_a^{(n)}(x) e^{i\omega_n t} \right), \tag{48}$$

where a_n and $a_n^\dagger$ satisfy the usual creation/annihilation operator relations:

$$[a_n, a_m^\dagger] = \delta_{nm}, \qquad [a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0. \tag{49}$$

This mode expansion is the foundation for computing quantum corrections to energies and for deriving coherence spectra.

5.5 Fluctuation Spectrum and Stability

The set $\{\omega_n^2\}$ constitutes the linearized spectrum around the background. Its properties determine:

- *stability*: If all $\omega_n^2 > 0$, the background is stable under small perturbations.
- *zero modes*: Modes with $\omega_n = 0$ correspond to symmetries of the solution (translations, rotations, etc.).
- *bound modes*: Discrete positive ω_n^2 indicate localized excitations near a soliton.
- *continuum*: The continuous part of the spectrum describes propagating waves in the vacuum.

In the context of coherence, bound modes represent "internal vibrations" of solitons, while continuum modes govern long–range transport of coherence disturbances.

5.6 Coherence Fluctuations

The effect of a fluctuation ξ^a on the coherence operator $\mathcal{C}(x)$ can be obtained by expanding

$$\mathcal{C}(x) = -\frac{f^2}{4}\mathrm{Tr}(L_i L_i) = \mathcal{C}^{(0)}(x) + \mathcal{C}^{(1)}(x) + \mathcal{C}^{(2)}(x) + \ldots, \tag{50}$$

where $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ describe the linear and quadratic corrections due to the fluctuation.

Using the mode expansion, one may evaluate

$$\langle 0|\,\hat{\mathcal{C}}(x)\,|0\rangle = \mathcal{C}^{(0)}(x) + \frac{1}{2}\sum_n \omega_n\,|\phi^{(n)}(x)|^2 + \cdots. \tag{51}$$

The second term represents the leading quantum coherence correction due to zero–point motion.

5.7 Coherence Kernel in the Linearised Regime

The general commutator of the renormalised coherence operator was defined in Eq. (31) as

$$[\hat{C}_R(t,\mathbf{x}), \hat{C}_R(t,\mathbf{y})] = i\hbar\, F(\mathbf{x},\mathbf{y}), \tag{52}$$

where $F(\mathbf{x},\mathbf{y})$ is a distribution fixed by the underlying current algebra. In this subsection we make this statement explicit in the regime of small quantum fluctuations around a classical background, using the linearised formalism developed above.

Let $U_0(x)$ be a classical solution of the Skyrme–Taiji equations of motion and consider small fluctuations as in Eq. (37),

$$U(x) = U_0(x)\, \exp\big(i\xi^a(x)\sigma_a\big), \tag{53}$$

with fluctuation operator D_{ab} and normal modes $\varphi_a^{(n)}(\mathbf{x})$ satisfying Eq. (42),

$$D_{ab}\, \varphi_b^{(n)}(\mathbf{x}) = \omega_n^2\, \varphi_a^{(n)}(\mathbf{x}). \tag{54}$$

The fluctuation field admits the mode expansion (43),

$$\xi^a(t,\mathbf{x}) = \sum_n \Big(a_n\, \varphi_a^{(n)}(\mathbf{x})\, e^{-i\omega_n t} + a_n^\dagger\, \varphi_a^{(n)}(\mathbf{x})\, e^{+i\omega_n t} \Big), \tag{55}$$

with canonical commutators

$$[a_n, a_m^\dagger] = \delta_{nm}, \qquad [a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0. \tag{56}$$

At the classical level, the coherence density is given by Eq. (28),

$$C(x) = -\frac{f^2}{4}\, \mathrm{Tr}\big(L_i L_i\big), \qquad i = 1,2,3. \tag{57}$$

Expanding the currents as

$$L_\mu(x) = L_\mu^{(0)}(x) + \delta L_\mu(x) + \mathcal{O}(\xi^2), \tag{58}$$

with $L_\mu^{(0)}$ the background currents and δL_μ linear in ξ^a, and promoting ξ^a to operators yields, after renormalisation, a mode expansion for $\hat{C}_R$ of the schematic form

$$\hat{C}_R(t,\mathbf{x}) = C^{(0)}(\mathbf{x}) + \sum_n \Big(b_n\, f_n(\mathbf{x})\, e^{-i\omega_n t} + b_n^\dagger\, f_n^*(\mathbf{x})\, e^{+i\omega_n t} \Big) + \cdots, \tag{59}$$

where:

- $C^{(0)}(\mathbf{x})$ is the classical coherence density of the background configuration,
- $f_n(\mathbf{x})$ are complex mode functions obtained by inserting the fluctuation expansion into the composite operator definition and collecting the terms linear in a_n and $a_n^\dagger$,
- b_n and $b_n^\dagger$ are background–dependent linear combinations of a_n and $a_n^\dagger$ chosen such that

$$[b_n, b_m^\dagger] = \delta_{nm}, \qquad [b_n, b_m] = [b_n^\dagger, b_m^\dagger] = 0. \tag{60}$$

The ellipsis in Eq. (59) denotes normal–ordered quadratic and higher contributions, which do not modify the leading equal–time commutator structure.

Using Eqs. (59) and (60), the equal–time commutator at time t becomes

$$\begin{aligned}[\hat{C}_R(t,\mathbf{x}), \hat{C}_R(t,\mathbf{y})] &= \sum_{n,m} \Big[b_n f_n(\mathbf{x}) + b_n^\dagger f_n^*(\mathbf{x}),\, b_m f_m(\mathbf{y}) + b_m^\dagger f_m^*(\mathbf{y}) \Big] \\ &= \sum_n \Big(f_n(\mathbf{x}) f_n^*(\mathbf{y}) - f_n(\mathbf{y}) f_n^*(\mathbf{x}) \Big),\end{aligned} \tag{61}$$

where we have used $[b_n, b_m^\dagger] = \delta_{nm}$ and suppressed an overall factor of $\hbar$ for compactness. Comparing with Eq. (52), we obtain the explicit mode–sum representation of the kernel in the linearised regime:

$$F(\mathbf{x},\mathbf{y}) = \frac{1}{i\hbar}\sum_n \Big(f_n(\mathbf{x}) f_n^*(\mathbf{y}) - f_n(\mathbf{y}) f_n^*(\mathbf{x}) \Big). \tag{62}$$

Thus, in the linearised theory the coherence kernel is completely determined by the fluctuation spectrum $\{\omega_n\}$ and the associated mode functions $\{f_n(\mathbf{x})\}$ constructed from the underlying Skyrme–Taiji dynamics.

5.7.1 Translationally invariant vacuum

A particularly transparent case is the translationally invariant vacuum, for which the background field is spatially constant,

$$U_0(x) = \mathbf{1}, \tag{63}$$

and the fluctuation operator D_{ab} is invariant under spatial translations. In this regime, the normal modes form a continuum labelled by spatial momentum $\mathbf{k}$ (and any internal index λ associated with the SU(2) structure). The index n is replaced by $(\lambda, \mathbf{k})$, and the mode functions may be chosen as plane waves,

$$f_{\lambda,\mathbf{k}}(\mathbf{x}) = f_\lambda(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{64}$$

with amplitudes $f_\lambda(\mathbf{k})$ determined by the composite operator structure of $\hat{C}_R$. The mode sum in Eq. (62) becomes a sum over λ and an integral over momenta,

$$F(\mathbf{x},\mathbf{y}) = \frac{1}{i\hbar}\sum_\lambda \int \frac{d^3k}{(2\pi)^3} \Big(f_\lambda(\mathbf{k}) f_\lambda^*(\mathbf{k})\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} - f_\lambda(\mathbf{k}) f_\lambda^*(\mathbf{k})\, e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})} \Big). \tag{65}$$

Defining the positive weight function

$$W(\mathbf{k}) \equiv \sum_\lambda |f_\lambda(\mathbf{k})|^2, \tag{66}$$

we obtain

$$F(\mathbf{x},\mathbf{y}) = \frac{2}{\hbar}\int \frac{d^3k}{(2\pi)^3}\, W(\mathbf{k})\, \sin\big(\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})\big). \tag{67}$$

In the translationally invariant vacuum, the coherence kernel thus depends only on the spatial separation $\mathbf{x}-\mathbf{y}$, as expected:

$$F(\mathbf{x},\mathbf{y}) = F(\mathbf{x}-\mathbf{y}). \tag{68}$$

The detailed form of $W(\mathbf{k})$ encodes the composite nature of $\hat{C}_R$, the dispersion relation of the underlying modes and the renormalisation scheme. In principle, it can be computed explicitly by inserting the free–field plane–wave expansion of the currents into the definition of $\hat{C}_R$ and matching coefficients; this is technically involved but does not affect the structural conclusions.

For equal times, Eq. (67) shows that:

- the kernel is antisymmetric under $\mathbf{x} \leftrightarrow \mathbf{y}$,
- $F(\mathbf{x},\mathbf{y})$ vanishes when averaged over regions much larger than the inverse cutoff scale if $W(\mathbf{k})$ decays sufficiently fast at large $|\mathbf{k}|$,
- the nontrivial commutator structure is concentrated in the short–distance behaviour, as required by locality and the EFT character of the theory.

These features feed directly into the coherence uncertainty relation (34), which, when applied to smeared operators $\hat{C}_R[f]$ and $\hat{C}_R[g]$, yields a concrete lower bound on the product of coherence variances in overlapping spatial regions, determined by the smeared kernel constructed from Eq. (67).

We emphasise that Eq. (62) and its specialisation (67) are derived within the linearised regime. For strongly non-linear backgrounds such as solitons, the same mode–sum structure applies in principle, with discrete bound modes and a modified continuum, but the explicit evaluation of $F(\mathbf{x}, \mathbf{y})$ requires a separate spectral analysis of the fluctuation operator around the chosen background and is left for future work.

5.8 Physical Interpretation

Linearized fluctuations reveal how coherence behaves quantum mechanically:

- They determine how stable a soliton is under quantum perturbations.
- They quantify the minimal coherence fluctuations required by the uncertainty principle.
- They reveal the normal modes that mediate the exchange of coherence between regions.
- They set the structure of entanglement modes in the next sections.

In this sense, the linearized theory provides the "first quantum correction layer" above the classical Skyrme–Taiji model. Subsequent sections will use these results to construct coherence spectra, path integrals and entanglement measures.

5.9 Photon Sector of the Coherence Connection

The quantized Skyrme–Taiji field provides the non–linear, topological backbone of the theory: its excitations include solitons, internal vibrational modes, and continuum coherence waves, all encoded in the fluctuation spectrum of the SU(2) field $U(x)$ and the associated coherence operator $\hat{C}_R(x)$. In the full Taiji–Skyrme framework, this structure is supplemented by an internal U(1) gauge degree of freedom associated with the local phase of the Taiji spinor, described at the classical level by the coherence connection $A_\mu(x)$ and its field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{69}$$

with dynamics governed by the Maxwell Lagrangian

$$\mathcal{L}_{\mathrm{EM}} = -\frac{1}{4g_{\mathrm{EM}}^2} F_{\mu\nu} F^{\mu\nu}. \tag{70}$$

This is the same A_μ that appears in the classical Maxwell–extended Skyrme–Taiji Lagrangian; no second electromagnetic field is introduced.

At the quantum level, the Maxwell sector is quantized in the standard way. After a choice of gauge and implementation of Gauss's law, one is left with two transverse photon polarizations. In a momentum–space basis one may write the spatial components of the gauge field schematically as

$$A_i(x) = \sum_{\lambda=1,2} \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}}}} \left[b_{\mathbf{k},\lambda}\, \varepsilon_i^{(\lambda)}(\mathbf{k})\, e^{-ik\cdot x} + b_{\mathbf{k},\lambda}^\dagger\, \varepsilon_i^{(\lambda)}(\mathbf{k})\, e^{+ik\cdot x} \right], \tag{71}$$

with $k^0 = \omega_{\mathbf{k}} = |\mathbf{k}|$, polarization vectors $\varepsilon_i^{(\lambda)}(\mathbf{k})$ satisfying $k_i \varepsilon_i^{(\lambda)}(\mathbf{k}) = 0$, and creation/annihilation operators obeying the usual commutation relations

$$\left[b_{\mathbf{k},\lambda}, b_{\mathbf{k}',\lambda'}^\dagger \right] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \delta_{\lambda\lambda'}, \qquad \left[b_{\mathbf{k},\lambda}, b_{\mathbf{k}',\lambda'} \right] = \left[b_{\mathbf{k},\lambda}^\dagger, b_{\mathbf{k}',\lambda'}^\dagger \right] = 0. \tag{72}$$

These quanta are identified with photons: linear U(1) gauge excitations of the same coherence connection that tracks the internal Taiji–spinor phase.

Within the quantum Taiji–Skyrme framework, the excitation content may therefore be organised as follows:

- *Topological excitations:* Skyrmions (Taiji knots) and their multi–soliton configurations, whose quantum statistics are fixed by Finkelstein–Rubinstein quantization and whose internal structure is probed by the fluctuation operator spectrum.
- *Coherence excitations:* Bound and continuum modes of the SU(2) field around a given background, which determine the coherence spectrum, coherence fluctuations, and the structure of entanglement channels.
- *Photon modes:* Transverse U(1) gauge excitations of A_μ, propagating on the same spacetime and coupled to matter through the standard Maxwell energy–momentum tensor. They represent oscillations of the internal coherence phase, complementary to the SU(2) orientation encoded in $U(x)$.

In regimes where electromagnetic energy is dynamically important (for example, in a radiation bath or in strongly magnetised environments), the photon sector must be included explicitly, and its contribution to the total energy–momentum tensor is tracked via $T_{\mu\nu}^{(\mathrm{EM})}$. In many of the applications considered in the coherence phenomenology documents (e.g. neutral, weakly magnetised solitons or galaxy–scale rotation curves), electromagnetic effects are subdominant in the energy budget; in those cases, the photon sector can be treated as a decoupled background without altering the quantum coherence structure developed in this work.

The key point is that photons arise as ordinary U(1) gauge quanta of the coherence connection A_μ, sitting alongside Skyrmions and coherence modes in the full quantum spectrum of the Taiji–Skyrme framework. The present document focuses on the SU(2) sector and the quantum operator structure of coherence; the detailed interplay between photon excitations and coherence observables is left for future, more specialised studies.

6 Path–Integral Formulation of Coherence

6.1 Motivation and Role of the Functional Integral

While canonical quantization provides a clear operator picture of coherence, many questions are more naturally addressed using the path–integral formalism. In particular:

- the derivation of effective actions for coarse–grained coherence,
- the study of decoherence and influence functionals,
- the computation of nonperturbative transitions between topological sectors,
- the analysis of entanglement in spacetime regions,

are most transparently handled in a functional integral setting.

In the Skyrme–Taiji framework, the basic dynamical field is the $SU(2)$–valued field $U(x)$. The path integral sums over all field histories weighted by the classical action. Coherence observables, defined in Section 4, appear as composite insertions in this functional integral. This section organizes these ideas and introduces the central objects: the partition functional, generating functionals for coherence, and the coherence effective action.

6.2 Partition Functional and Topological Sectors

The starting point is the partition functional

$$\mathcal{Z} = \int_{\mathcal{C}} \mathcal{D}U \, \exp\!\left(\frac{i}{\hbar} S[U]\right), \tag{73}$$

where the action is

$$S[U] = \int \mathrm{d}^4x \, \mathcal{L}[U], \tag{74}$$

with $\mathcal{L}$ the Skyrme–Taiji Lagrangian of Section 2. The functional integration runs over the full configuration space $\mathcal{C}$, which decomposes into topological sectors,

$$\mathcal{C} = \bigcup_{Q \in \mathbb{Z}} \mathcal{C}_Q. \tag{75}$$

Accordingly, the partition functional may be written as a sum

$$\mathcal{Z} = \sum_{Q \in \mathbb{Z}} \mathcal{Z}_Q, \qquad \mathcal{Z}_Q = \int_{\mathcal{C}_Q} \mathcal{D}U \, \exp\!\left(\frac{i}{\hbar} S[U]\right). \tag{76}$$

The Finkelstein–Rubinstein quantization condition (Section 3.6) is implemented in this context by assigning appropriate phase factors to field histories corresponding to nontrivial loops in $\mathcal{C}_Q$.

6.3 Generating Functional for Coherence

To probe coherence, we couple an external source $J(x)$ to the renormalized coherence density $\mathcal{C}_R(x)$:

$$\mathcal{Z}[J] = \int_{\mathcal{C}} \mathcal{D}U \, \exp\!\left[\frac{i}{\hbar}\left(S[U] + \int \mathrm{d}^4x \, J(x)\, \mathcal{C}_R(x)\right)\right]. \tag{77}$$

Functional derivatives of $\mathcal{Z}[J]$ with respect to $J(x)$ generate coherence correlation functions:

$$\left.\frac{\delta \mathcal{Z}[J]}{\delta J(x)}\right|_{J=0} = \frac{i}{\hbar} \, \langle \mathcal{C}_R(x) \rangle \, \mathcal{Z}[0], \tag{78}$$

$$\left.\frac{\delta^2 \mathcal{Z}[J]}{\delta J(x)\, \delta J(y)}\right|_{J=0} = \left(\frac{i}{\hbar}\right)^2 \langle \mathcal{C}_R(x)\, \mathcal{C}_R(y) \rangle_c \mathcal{Z}[0], \tag{79}$$

and so forth, where the subscript c denotes connected correlators.

It is often convenient to work with the generating functional of connected diagrams,

$$W[J] = -i\hbar \ln \mathcal{Z}[J], \tag{80}$$

which satisfies

$$\frac{\delta W[J]}{\delta J(x)} = \langle \mathcal{C}_R(x) \rangle_J. \tag{81}$$

Here $\langle \cdot \rangle_J$ denotes expectation values in the presence of the source J.

6.4 Coarse–Grained Coherence Field and Effective Action

One can define a coarse–grained coherence field $\mathcal{C}(x)$ as the expectation value

$$\mathcal{C}(x) = \langle \mathcal{C}_R(x) \rangle_J, \tag{82}$$

and view $J(x)$ and $\mathcal{C}(x)$ as conjugate variables in the Legendre sense. The effective action for coherence is then obtained by a Legendre transform:

$$\Gamma[\mathcal{C}] = W[J] - \int \mathrm{d}^4x \, J(x)\, \mathcal{C}(x), \tag{83}$$

where J is eliminated in favor of $\mathcal{C}$ using

$$\frac{\delta\Gamma[\mathcal{C}]}{\delta\mathcal{C}(x)} = -J(x). \tag{84}$$

In the absence of external sources ($J = 0$), stationary configurations of Γ satisfy

$$\frac{\delta\Gamma[\mathcal{C}]}{\delta\mathcal{C}(x)} = 0, \tag{85}$$

and describe dynamically preferred coherence profiles.

In practice, $\Gamma[\mathcal{C}]$ encodes:

- the effective dynamics of long–wavelength coherence modes,
- the induced interactions between coherence domains,
- the quantum corrections to classical coherence patterns,
- potential instabilities leading to coherence phase transitions.

6.5 Decoherence and Influence Functionals

To describe decoherence, it is useful to consider the influence functional obtained by integrating out some degrees of freedom while keeping the coherence sector explicit. For instance, one may split the field into "system" and "environment" modes,

$$U(x) \to \big(U_{\mathrm{sys}}(x), U_{\mathrm{env}}(x)\big), \tag{86}$$

and integrate over U_{env}. This yields an effective nonlocal action for U_{sys}, or equivalently for coarse–grained coherence variables.

In the closed time path formalism, one introduces forward and backward field histories and defines the decoherence functional,

$$\mathcal{D}[U^+, U^-] = \exp\left(\frac{i}{\hbar} S_{\mathrm{eff}}[U^+, U^-]\right), \tag{87}$$

where S_{eff} includes both real and imaginary parts. The imaginary part governs the suppression of off–diagonal components in the reduced density matrix and thus quantifies decoherence.

In the Skyrme–Taiji context, this formalism makes it possible to ask:

- how quickly coherence between different soliton configurations is lost,
- how topological sectors mix under environmental influences,
- how macroscopic coherence emerges from microscopic fluctuations.

6.6 Coherence Kernels and Nonlocality

The path–integral formulation also provides a natural setting for coherence kernels. Expanding $W[J]$ to quadratic order in J,

$$W[J] = W[0] + \int \mathrm{d}^4x\, J(x)\, \mathcal{C}_0(x) - \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, J(x)\, K(x,y)\, J(y) + \cdots, \tag{88}$$

one identifies the kernel $K(x, y)$ as the inverse of the coherence propagator in a suitable sense. This kernel is generally nonlocal in space and time, reflecting the fact that coherence is not a strictly local quantity but involves correlations across regions.

Physically, $K(x, y)$ encodes:

- how perturbations of coherence at one point feed back into another,
- the "stiffness" of coherence patterns,
- the emergent coherence correlation length and timescale.

6.7 Summary and Outlook

The path–integral formulation complements the canonical framework by:

- providing a global view of quantum dynamics across topological sectors,
- defining generating functionals for coherence observables,
- enabling the construction of effective actions for coarse–grained coherence,
- giving a natural language for decoherence and influence functionals,
- clarifying the role of nonlocal coherence kernels.

In the Skyrme–Taiji model, these tools allow us to connect the microscopic topology and dynamics of the $SU(2)$ field to macroscopic patterns of order and decoherence. They establish coherence as a fully dynamical quantum degree of freedom, governed by its own effective action and kernels, rather than as a derived or phenomenological quantity.

The next step is to examine how this formalism constrains entanglement structures and coherence sharing between spatial regions and solitonic degrees of freedom.

7 Entanglement and Topological Coherence

7.1 Motivation and Conceptual Framework

Quantum coherence, as a local operator constructed from the $SU(2)$ currents, encodes how ordered or aligned the Skyrme–Taiji field is at each point. Quantum *entanglement*, in contrast, measures how correlations between spatial regions cannot be described classically. In the Taiji interpretation, both quantities play complementary roles:

- Coherence quantifies the local organization of the underlying order field.
- Entanglement quantifies the degree to which distant regions of the field share quantum information.

The interplay between these two notions is particularly rich in a topological field theory like the Skyrme–Taiji model. Because solitons correspond to nontrivial elements of $\pi_3(S^3)$, they carry global topological information. Quantum entanglement reveals how this information is distributed between regions, how it is shared, and how it evolves under dynamics.

7.2 Hilbert Space Factorization and Topological Sectors

In an ordinary quantum field theory, the Hilbert space associated with a spatial region Ω and its complement $\bar{\Omega}$ factorizes as

$$\mathcal{H} = \mathcal{H}_\Omega \otimes \mathcal{H}_{\bar{\Omega}}. \tag{89}$$

However, in topological field theories or theories with solitonic sectors, this factorization is subtle. Because field configurations belong to topological sectors $\mathcal{C}_Q$, cutting space into subregions may force field configurations on Ω and $\bar{\Omega}$ to share nonlocal constraints:

$$U(x) \in \mathcal{C}_Q \quad \Rightarrow \quad (U|_\Omega, U|_{\bar{\Omega}}) \text{ must be topologically compatible.} \tag{90}$$

This means that the Hilbert space decomposes as

$$\mathcal{H}_Q \neq \mathcal{H}_{Q,\Omega} \otimes \mathcal{H}_{Q,\bar{\Omega}}, \tag{91}$$

but instead requires a "gluing rule" that ensures continuity and topological matching across the boundary $\partial\Omega$.

7.3 Reduced Density Matrix and Entanglement Entropy

Given a quantum state $|\Psi\rangle$, the reduced density matrix for a region Ω is defined as

$$\rho_\Omega = \mathrm{Tr}_{\bar{\Omega}} |\Psi\rangle\langle\Psi|. \tag{92}$$

The entanglement entropy is

$$S_\Omega = -\mathrm{Tr}\left(\rho_\Omega \ln \rho_\Omega\right). \tag{93}$$

In the Skyrme–Taiji context, S_Ω measures:

- how much quantum information is shared between region Ω and its complement,
- how strongly coherence patterns in Ω depend on those in $\bar{\Omega}$,
- how soliton topology constrains entanglement,
- the "coherence capacity" of a region.

7.4 Entanglement of Soliton States

Consider a single soliton state $|Q\rangle$ with $Q = 1$. Because of Finkelstein–Rubinstein quantization, this state carries fermionic statistics. The topological charge forces $U(x)$ to wrap the vacuum manifold in a globally nontrivial way. This imposes nonlocal structure on the field configuration, and hence the entanglement entropy of spatial regions is sensitive to the soliton.

For example, if Ω contains the center of the soliton, then

$$S_\Omega^{(Q=1)} > S_\Omega^{(\mathrm{vac})}, \tag{94}$$

because the soliton induces both:

- local coherence distortions,
- global topological constraints.

These constraints give rise to "topologically protected" contributions to entanglement, originated from the nontrivial wrapping of the field.

7.5 Coherence Correlations and Entanglement

The two–point coherence correlator

$$G_{\mathcal{C}}(x, y) = \langle\, \hat{\mathcal{C}}_R(x)\, \hat{\mathcal{C}}_R(y)\, \rangle \tag{95}$$

controls how coherence at x is linked to coherence at y. In general,

$$G_{\mathcal{C}}(x, y) \neq 0 \quad \Rightarrow \quad S_\Omega \text{ increases if } x \in \Omega,\ y \in \bar{\Omega}. \tag{96}$$

Thus coherence transport (Section 6) and entanglement are deeply connected:

- low–frequency coherence modes contribute long–range entanglement,
- soliton zero modes (especially rotational modes governed by FR) generate characteristic entanglement patterns,
- bound coherence modes create localized contributions to entanglement.

Because the coherence operator is quadratic in the currents L_i, the entanglement entropy inherits contributions from fluctuations at all wavelengths.

7.6 Entanglement and FR Quantization

Finkelstein–Rubinstein quantization affects entanglement in two ways:

1. It imposes a sign change under 2π rotation for odd Q.
2. It restricts the allowed field histories in the path integral.

This means that the reduced density matrix includes only those histories consistent with the FR condition. As a result:

$$S_\Omega^{(Q\text{ odd})} \neq S_\Omega^{(Q\text{ even})}, \tag{97}$$

even if local energy densities are similar. This reflects a general physical principle:

> Topological constraints produce distinct entanglement patterns even in otherwise identical local environments.

7.7 Coherence Sharing and Quantum Communication Channels

The coherence spectrum determines which modes mediate entanglement between separated regions. In particular:

- Zero modes generate the longest–range entanglement channels.
- Bound modes confine entanglement near soliton cores.
- High–frequency modes provide only short–range correlations.

The "transport" of coherence through a field configuration therefore determines the "communication pathways" for quantum entanglement. This is particularly important when analyzing:

- soliton–soliton interactions,
- scattering processes,
- soliton bound states,
- topological transitions.

7.8 Summary and Outlook

Entanglement, as formulated here, is not an optional or secondary feature of the Skyrme–Taiji theory. It is a structural property forced by:

- the topology of the field,
- the nature of coherence as a composite operator,
- the fluctuation spectrum,
- the FR quantization rules.

Together, these establish a quantum–mechanical fabric in which coherence and entanglement are inseparably intertwined. Entanglement reveals how coherence patterns distribute across space and topological sectors, and it governs the stability and interactions of solitons at the quantum level.

This completes the quantum formulation of coherence. The next (optional) section may summarize the main results or develop applications such as soliton spectroscopy, coherence transport, or quantum gravitational corrections.

8 Summary and Conclusion

8.1 Overview of the Quantum Coherence Framework

This document has constructed a complete and internally consistent quantum formulation of coherence within the Skyrme–Taiji model. Beginning with the classical $SU(2)$ field and its topological structure, we developed the canonical quantization of the field, introduced the renormalized coherence operator, analyzed its fluctuation spectrum, and established the functional integral framework that governs coherence correlations, decoherence, and entanglement.

The resulting theory is structurally unified: each layer supports and constrains the next, producing a coherent whole rather than a collection of independent formalisms.

8.2 Main Results

The essential results of this work may be summarized as follows:

1. Canonical Structure. The phase space of the theory is built from the Maurer–Cartan currents, with Poisson brackets promoted to operator commutators. The resulting algebra preserves the geometrical structure of $SU(2)$ and provides the foundation for a consistent quantum description.

2. Finkelstein–Rubinstein Quantization. The nontrivial topology of the configuration space requires the Finkelstein–Rubinstein rule. As a consequence, solitonic excitations inherit definite quantum statistics: odd topological charge corresponds to fermionic behavior, even charge to bosonic behavior.

3. Renormalized Coherence Operator. The coherence density, constructed from local currents, becomes a composite operator in the quantum theory. Renormalization via point–splitting yields a finite, well–defined operator $\mathcal{C}_R(x)$ whose expectation values and correlation functions are physically meaningful.

4. Linearized Quantum Fluctuations. Small fluctuations around classical backgrounds give rise to a self–adjoint fluctuation operator whose eigenmodes determine:

- the stability of solitons,
- the internal excitation spectrum,
- the zero modes associated with symmetries,
- the ultraviolet structure of coherence fluctuations.

5. The Coherence Spectrum. The eigenfrequencies of the fluctuation operator form the coherence spectrum. This spectrum governs the transport, localization, and decay of coherence across space and plays a central role in the quantum structure of solitons.

6. Path Integral Formulation. The path integral over the group manifold $SU(2)$ organizes nonperturbative dynamics, enabling the construction of generating functionals, effective actions for coarse–grained coherence, and decoherence functionals based on integrating out environmental modes.

7. Entanglement and Topology. We established that quantum entanglement between spatial regions is strongly influenced by:

- topological charge,
- coherence correlations,
- the FR quantization rule,
- the solitonic structure of the field.

Entanglement thus becomes an intrinsic part of the quantum Taiji model, not an optional addition.

8.3 Conceptual Implications

The quantum coherence framework reveals several deep conceptual features:

1. Coherence behaves as a true quantum degree of freedom, possessing its own operator algebra and uncertainty relations.
2. The topological sector of a configuration dictates its quantum statistics and entanglement characteristics.
3. Quantum corrections, encoded in the coherence spectrum, provide both stability and internal structure to solitons.
4. Nonlocal features of coherence, visible in correlation kernels and reduced density matrices, reflect the fundamentally interconnected nature of the $SU(2)$ field.

These features merge topology, geometry, and quantum mechanics into a single coherent structure.

8.4 Closing Perspective

The quantum theory of coherence presented here is not merely a technical extension of the classical Skyrme–Taiji model. It establishes coherence as a fundamental organizing principle at the quantum level, revealing:

- how order emerges and persists,
- how quantum solitons acquire structure and statistics,
- how regions of space share information,
- how topological constraints shape quantum behavior.

This framework forms the foundation for further investigations into quantum soliton spectroscopy, long–range coherence transport, effective collective dynamics, and potential links to emergent gravitational or cosmological phenomena.

It closes the formal structure of the quantum Skyrme–Taiji theory and prepares the ground for applications across the full Taiji–Skyrme paradigm.

9 Appendix A: Configuration Space Topology in the Taiji Spinor Representation

The purpose of this appendix is to show that the Taiji spinor representation does not alter the topological structure of the configuration space relevant for Finkelstein–Rubinstein quantization. In particular, we argue that the configuration space built from normalized Taiji spinors is homotopy–equivalent to the standard Skyrme configuration space, so that the fundamental group

$$\pi_1(\mathcal{C}_Q) \cong \begin{cases} \mathbb{Z}_2, & Q \text{ odd}, \\ 0, & Q \text{ even}, \end{cases}$$

and hence the FR rule used in Section 3.6 remains valid in the full Taiji–Skyrme setting.

A.1 Classical Skyrme Configuration Space

For finite–energy field configurations, the SU(2)–valued Skyrme field $U(x)$ must tend to a constant group element at spatial infinity,

$$U(x) \xrightarrow{|x|\to\infty} U_\infty,$$

which may be taken, without loss of generality, to be the identity. Identifying all points at spatial infinity, physical space is compactified to S^3 and the field configurations are continuous maps

$$U : S^3_{\text{space}} \to S^3_{\text{group}} \simeq SU(2).$$

These maps fall into homotopy classes labelled by the winding number $Q \in \mathbb{Z} \cong \pi_3(S^3)$, and the full configuration space decomposes as

$$\mathcal{C} = \bigcup_{Q\in\mathbb{Z}} \mathcal{C}_Q,$$

with $\mathcal{C}_Q$ the connected component of fixed topological charge. For each sector $\mathcal{C}_Q$, the fundamental group $\pi_1(\mathcal{C}_Q)$ determines the possible nontrivial loops in configuration space and underlies the Finkelstein–Rubinstein quantization condition used in Section 3.6.

A.2 Normalized Taiji Spinor and SU(2) Map

In the Taiji–Skyrme framework, the fundamental field is a normalized Taiji spinor $Z(x)$ encoding the Excitation–Potential Field structure. The normalization condition

$$Z^\dagger(x)Z(x) = 1$$

implies that at each spacetime point x the spinor $Z(x)$ lies on a three–sphere S^3_Z. The vacuum manifold discussed in Section 2.2 is therefore

$$\mathcal{M}_{\text{vac}} \simeq S^3_Z,$$

with nontrivial homotopy $\pi_3(\mathcal{M}_{\text{vac}}) \cong \mathbb{Z}$ inherited from the three–sphere structure.

Section 2.3 introduced an explicit map from the normalized spinor $Z(x)$ to an SU(2) group element $U(x)$, schematically of the form

$$U[Z(x)] \in SU(2),$$

which is smooth and bijective at each point and respects the group manifold geometry. As a consequence, the vacuum manifold described by Z and the group manifold described by U are smoothly equivalent:

$$S^3_Z \simeq S^3_{\text{group}} \simeq SU(2).$$

A.3 Homotopy Equivalence of Configuration Spaces

Finite–energy conditions on $Z(x)$ require

$$Z(x) \xrightarrow{|x|\to\infty} Z_\infty,$$

with Z_∞ a constant normalized spinor corresponding to the vacuum. After compactification of space to S^3_{space}, a configuration of the Taiji spinor field is therefore a continuous map

$$Z : S^3_{\text{space}} \to S^3_Z.$$

Composing this map with the pointwise diffeomorphism $Z \mapsto U[Z]$ yields a corresponding SU(2)–valued configuration,

$$U[Z] : S^3_{\text{space}} \to S^3_{\text{group}},$$

which satisfies the same finite–energy boundary condition at spatial infinity. Conversely, given an SU(2) configuration $U(x)$, the inverse of the map of Section 2.3 reconstructs a normalized spinor configuration $Z(x)$.

This establishes a one–to–one correspondence between:

- homotopy classes of spinor configurations $Z : S^3_{\text{space}} \to S^3_Z$,
- homotopy classes of SU(2) configurations $U : S^3_{\text{space}} \to S^3_{\text{group}}$.

In particular, the topological charge defined via the SU(2) field, as in the standard Skyrme model, coincides with the winding of the spinor map. The configuration space sectors $\mathcal{C}_Q$ defined in terms of $U(x)$ are therefore in homotopy–equivalent correspondence with the sectors defined in terms of $Z(x)$. The inclusion of the Taiji spinor does *not* change the decomposition

$$\mathcal{C} = \bigcup_{Q\in\mathbb{Z}} \mathcal{C}_Q,$$

nor the homotopy groups of each sector.

A.4 Fundamental Group and FR Quantization

Because the configuration space sectors $\mathcal{C}_Q$ in the spinor representation are homotopy–equivalent to those of the standard SU(2) Skyrme model, the fundamental group retains its known structure:

$$\pi_1(\mathcal{C}_Q) \cong \begin{cases} \mathbb{Z}_2, & Q \text{ odd}, \\ 0, & Q \text{ even}. \end{cases}$$

The nontrivial element for odd Q corresponds to a loop in configuration space implementing a 2π spatial rotation of the entire field configuration. As shown in Section 3.6, consistency of the quantum theory requires wavefunctionals to pick up a phase

$$\Psi[U_{2\pi}] = e^{i\alpha}\Psi[U_0], \qquad e^{i\alpha} = (-1)^Q,$$

so that odd–Q solitons are quantized as fermions and even–Q solitons as bosons.

The Taiji spinor representation therefore preserves the FR quantization rule and the associated spin–statistics connection. No additional Wess–Zumino–Witten term or higher–dimensional extension is required to obtain fermionic solitons: the topology of the configuration space, inherited from the mapping $S^3_{\text{space}} \to S^3_Z \simeq SU(2)$, is sufficient to enforce the FR condition used throughout the quantum coherence framework.

10 Appendix B: One-loop vacuum effective potential for the coherence field

In this appendix we derive the structure of the effective potential $V_{\rm eff}(\psi)$ for the classical coherence order parameter ψ in a controlled quantum regime of the Taiji–Skyrme theory. The goal is to show that, near a homogeneous vacuum and at one-loop order, the microscopic Skyrme sector naturally induces a Ginzburg–Landau type polynomial in ψ up to quartic order.

10.1 Setup and assumptions

We work in flat Minkowski spacetime with metric signature $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ and ignore gravitational backreaction. The relevant fields are:

1. The Skyrme field $U(x) \in SU(2)$ with standard kinetic and Skyrme terms, as defined in the main text.
2. The Taiji spinor $Z(x) \in \mathbb{C}^2$, whose norm $C(x) = |Z(x)|^2$ defines the local coherence amplitude.
3. The dimensionless coherence order parameter

$$\psi(x) \equiv \frac{C(x)}{v^2} = \frac{|Z(x)|^2}{v^2}\,, \tag{98}$$

   where $v > 0$ is a fixed coherence normalisation scale.

We consider homogeneous background configurations of the form

$$U(x) = U_0 = \mathbb{I}\,, \qquad \psi(x) = \psi_0 = \text{const.} \tag{99}$$

and integrate out small quantum fluctuations of the Skyrme field around U_0 in the presence of a constant ψ_0. In this regime we assume:

- ψ varies slowly compared to the microscopic Skyrme scale, so that derivative terms in ψ can be treated within a derivative expansion of the effective action.
- The Skyrme fluctuations are weakly coupled, so that a loop expansion is meaningful and truncation at one loop is consistent.

We do not fix numerical values of coupling constants or cutoffs in this appendix; all expressions are kept at the level of symbolic dependence on the Taiji–Skyrme parameters.

10.2 Tree-level Lagrangian and mass spectrum

The classical Skyrme Lagrangian in flat space has the form

$$\mathcal{L}_{\rm Sky} = \frac{f^2}{4}\,\mathrm{Tr}(L_\mu L^\mu) + \frac{1}{32e^2}\mathrm{Tr}([L_\mu, L_\nu][L^\mu, L^\nu]) - V_{\rm Sky}(U)\,, \tag{100}$$

where $L_\mu = U^\dagger \partial_\mu U$, f is the Skyrme stiffness scale, e is a dimensionless coupling, and $V_{\rm Sky}(U)$ is a symmetry-breaking potential fixing the vacuum manifold. A typical choice is

$$V_{\rm Sky}(U) \sim m^2 f^2\,\mathrm{Tr}(\mathbb{I} - U)\,, \tag{101}$$

which gives a mass scale m to the pion fluctuations.

The coherence order parameter ψ couples to the Skyrme sector through coherence-dependent parameters and/or explicit interaction terms. At the level of a local effective Lagrangian we can write

$$\mathcal{L}_{\rm coh} = \frac{1}{2}(\partial_\mu\psi)(\partial^\mu\psi) - V_{\rm tree}(\psi) - \mathcal{L}_{\rm int}[U,\psi]\,, \tag{102}$$

where $V_{\rm tree}(\psi)$ denotes any tree-level potential in the coherence sector and $\mathcal{L}_{\rm int}[U,\psi]$ encodes ψ-dependent couplings to the Skyrme field. The precise form of $\mathcal{L}_{\rm int}$ depends on the chosen microscopic implementation; for the present purpose it suffices that, after expanding around $U_0 = \mathbb{I}$ and $\psi = \psi_0$, the pion modes acquire a ψ_0-dependent mass.

Expanding U in pion fields $\pi^a(x)$ via

$$U(x) = \exp\!\left(\frac{i\pi^a(x)\sigma_a}{f}\right), \tag{103}$$

one obtains a quadratic fluctuation Lagrangian of the form

$$\mathcal{L}_{\rm quad} = \frac{1}{2}Z_\pi(\psi_0)\,\partial_\mu\pi^a\partial^\mu\pi^a - \frac{1}{2}M^2(\psi_0)\,\pi^a\pi^a + \dots, \tag{104}$$

where $Z_\pi(\psi_0)$ is a field-strength renormalisation factor and $M^2(\psi_0)$ is the effective pion mass-squared in the coherence background ψ_0. After canonical normalisation of the pion fields, the physical mass-squared can be written as

$$M^2(\psi_0) = M_0^2 + \alpha_1(\psi_0 - \bar\psi) + \alpha_2(\psi_0 - \bar\psi)^2 + \dots, \tag{105}$$

where M_0^2 is the mass-squared at some reference coherence $\bar\psi$ and $\alpha_1, \alpha_2, \dots$ are functions of the microscopic parameters (f, e, m) and the coherence-normalisation constants $(\alpha_C, \dots)$.

10.3 One-loop effective potential

The one-loop contribution to the effective potential from N_π degenerate scalar modes of mass $M(\psi_0)$ is given, in dimensional regularisation and $\overline{\rm MS}$ scheme, by the standard Coleman–Weinberg expression

$$V^{(1)}(\psi_0) = \frac{N_\pi}{64\pi^2}M^4(\psi_0)\left[\ln\!\left(\frac{M^2(\psi_0)}{\mu^2}\right) - \frac{3}{2}\right], \tag{106}$$

where μ is the renormalisation scale. In the Taiji–Skyrme model we have $N_\pi = 3$ pion degrees of freedom, but we keep N_π as a symbolic parameter.

To obtain the dependence of $V^{(1)}$ on the coherence field, we substitute the expansion (105) into (106). Writing

$$\Delta\psi \equiv \psi_0 - \bar\psi, \tag{107}$$

we expand $M^2(\psi_0)$ and $M^4(\psi_0)$ as

$$M^2(\psi_0) = M_0^2 + \alpha_1\Delta\psi + \alpha_2\Delta\psi^2 + \mathcal{O}(\Delta\psi^3), \tag{108}$$

$$M^4(\psi_0) = M_0^4 + 2M_0^2\alpha_1\Delta\psi + \big(\alpha_1^2 + 2M_0^2\alpha_2\big)\Delta\psi^2 + \mathcal{O}(\Delta\psi^3), \tag{109}$$

and expand the logarithm as

$$\ln\!\left(\frac{M^2(\psi_0)}{\mu^2}\right) = \ln\!\left(\frac{M_0^2}{\mu^2}\right) + \frac{\alpha_1}{M_0^2}\Delta\psi + \left(\frac{\alpha_2}{M_0^2} - \frac{\alpha_1^2}{2M_0^4}\right)\Delta\psi^2 + \mathcal{O}(\Delta\psi^3). \tag{110}$$

Substituting these series into (106) and collecting powers of $\Delta\psi$ yields an expansion

$$V^{(1)}(\psi_0) = V^{(1)}(\bar\psi) + A_1\Delta\psi + \frac{1}{2}A_2\Delta\psi^2 + \frac{1}{3!}A_3\Delta\psi^3 + \frac{1}{4!}A_4\Delta\psi^4 + \mathcal{O}(\Delta\psi^5), \tag{111}$$

with coefficients A_n that are explicit functions of $(N_\pi, M_0, \alpha_1, \alpha_2, \mu)$ and hence of the underlying Taiji–Skyrme parameters (f, e, m) and coherence couplings. The constant term $V^{(1)}(\bar\psi)$ can be absorbed into a redefinition of the vacuum energy and will be dropped in what follows.

10.4 Truncated effective potential and Ginzburg–Landau form

Including any tree-level potential $V_{\text{tree}}(\psi)$, the total effective potential up to one loop can be expanded around $\psi = \bar{\psi}$ as

$$V_{\text{eff}}(\psi) = V_{\text{tree}}(\bar{\psi}) + \tilde{A}_1 \Delta\psi + \frac{1}{2}\tilde{A}_2 \Delta\psi^2 + \frac{1}{3!}\tilde{A}_3 \Delta\psi^3 + \frac{1}{4!}\tilde{A}_4 \Delta\psi^4 + \mathcal{O}(\Delta\psi^5), \tag{112}$$

where $\tilde{A}_n$ are the sum of tree-level and one-loop contributions. Choosing $\bar{\psi}$ to be an extremum of V_{eff} (so that $\tilde{A}_1 = 0$), we obtain

$$V_{\text{eff}}(\psi) = V_0 + \frac{1}{2}\tilde{A}_2 \Delta\psi^2 + \frac{1}{3!}\tilde{A}_3 \Delta\psi^3 + \frac{1}{4!}\tilde{A}_4 \Delta\psi^4 + \mathcal{O}(\Delta\psi^5), \tag{113}$$

with $V_0 = V_{\text{eff}}(\bar{\psi})$.

In practical applications we are interested in the regime where $\Delta\psi$ is small and higher-order terms can be neglected. In this regime we can truncate (113) at quartic order and write

$$V_{\text{eff}}(\psi) \simeq V_0 + \frac{1}{2}\tilde{A}_2 \Delta\psi^2 + \frac{1}{3!}\tilde{A}_3 \Delta\psi^3 + \frac{1}{4!}\tilde{A}_4 \Delta\psi^4. \tag{114}$$

This is precisely of Ginzburg–Landau type: a polynomial in $(\psi - \bar{\psi})$ with coefficients determined by the microscopic Taiji–Skyrme parameters and the chosen renormalisation scheme. The signs and relative magnitudes of $\tilde{A}_2$ and $\tilde{A}_4$ decide the local structure of the coherence vacuum:

- If $\tilde{A}_2 > 0$ and $\tilde{A}_4 > 0$, then $\psi = \bar{\psi}$ is a local minimum with single-well structure near $\bar{\psi}$.
- If $\tilde{A}_2 < 0$ and $\tilde{A}_4 > 0$, then V_{eff} exhibits a double-well structure in ψ, with two symmetry-related minima separated by a barrier in the neighbourhood of $\bar{\psi}$.

More general multi-branch structures arise when $\tilde{A}_3 \neq 0$ or when higher-order terms beyond quartic are retained, but the quartic truncation (114) already captures the basic scenario of multiple coherence vacua separated by domain walls.

10.5 Comments on higher-loop corrections

Higher-loop corrections modify the effective potential by adding further contributions $V^{(2)}(\psi), V^{(3)}(\psi), \ldots$ built from two-loop and higher vacuum diagrams with ψ-dependent masses and vertices. In a neighbourhood of a chosen vacuum $\bar{\psi}$, these corrections can again be expanded in powers of $\Delta\psi$ and absorbed into renormalised coefficients $\tilde{A}_n \to \tilde{A}_n^{(\text{ren})}$ in (113). As long as the loop expansion remains perturbative, the local polynomial structure of $V_{\text{eff}}(\psi)$ is preserved and the qualitative features of the coherence vacuum structure (single well vs. double well, barrier height, etc.) remain controlled by the signs and magnitudes of the renormalised coefficients.

A complete determination of the numerical values of $\tilde{A}_2, \tilde{A}_3, \tilde{A}_4$ requires a specific choice of microscopic parameters $(f, e, m, \ldots)$, an explicit form of $\mathcal{L}_{\text{int}}[U, \psi]$, and a renormalisation scheme for the loop integrals. In this appendix we restrict ourselves to the structural result: in the one-loop vacuum regime of the Taiji–Skyrme theory, the coherence order parameter ψ acquires an effective potential of Ginzburg–Landau type, with a quartic truncation of the form (114) whose coefficients are computable functions of the underlying Taiji–Skyrme parameters.